

# Induced minors and well-quasi-ordering\*

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## Abstract

A graph  $H$  is an induced minor of a graph  $G$  if it can be obtained from an induced subgraph of  $G$  by contracting edges. Otherwise,  $G$  is said to be  $H$ -induced minor-free. Robin Thomas showed that  $K_4$ -induced minor-free graphs are well-quasi-ordered by induced minors [*Graphs without  $K_4$  and well-quasi-ordering*, Journal of Combinatorial Theory, Series B, 38(3):240 – 247, 1985].

We provide a dichotomy theorem for  $H$ -induced minor-free graphs and show that the class of  $H$ -induced minor-free graphs is well-quasi-ordered by the induced minor relation if and only if  $H$  is an induced minor of the Gem (the path on 4 vertices plus a dominating vertex) or of the graph obtained by adding a vertex of degree 2 to the complete graph on 4 vertices. To this end we proved two decomposition theorems which are of independent interest.

Similar dichotomy results were previously given for subgraphs by Guoli Ding in [*Subgraphs and well-quasi-ordering*, Journal of Graph Theory, 16(5):489–502, 1992] and for induced subgraphs by Peter Damaschke in [*Induced subgraphs and well-quasi-ordering*, Journal of Graph Theory, 14(4):427–435, 1990].

## 1 Introduction

A *well-quasi-order* (*wqo* for short) is a quasi-order which contains no infinite decreasing sequence, nor an infinite collection of pairwise incomparable elements (called an *antichain*). One of the most important results in this field is arguably the theorem by Robertson and Seymour which states that graphs are well-quasi-ordered by

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the minor relation [RS04]. Other natural containment relations are not so generous; they usually do not wqo all graphs. In the last decades, much attention has been brought to the following question: given a partial order  $(S, \preceq)$ , what subclasses of  $S$  are well-quasi-ordered by  $\preceq$ ? For instance, Fellows et al. proved in [FHR09] that graphs with bounded feedback-vertex-set are well-quasi-ordered by topological minors. Another result is that of Oum [iO08] who proved that graphs of bounded rank-width are wqo by vertex-minors. Other papers considering this question include [Dam90, Tho85, Din92, KRT14, HL14, Din98, Din09, AL14, Pet02, DRT10].

One way to approach this problem is to consider graph classes defined by excluded substructures. In this direction, Damaschke proved in [Dam90] that a class of graphs defined by one forbidden induced subgraph  $H$  is wqo by the induced subgraph relation if and only if  $H$  is the path on four vertices. Similarly, a bit later Ding proved in [Din92] an analogous result for the subgraph relation. Other authors also considered this problem (see for instance [KL11a, KL11b, Che11]). In this paper, we provide an answer to the same question for the induced minor relation, which we denote  $\leq_{\text{im}}$ . Before stating our main result, let us introduce two graphs which play a major role in this paper (see Figure 1). The first one,  $\hat{K}_4$ , is obtained by adding a vertex of degree two to  $K_4$ , and the second one, called the Gem, is constructed by adding a dominating vertex to  $P_4$ .

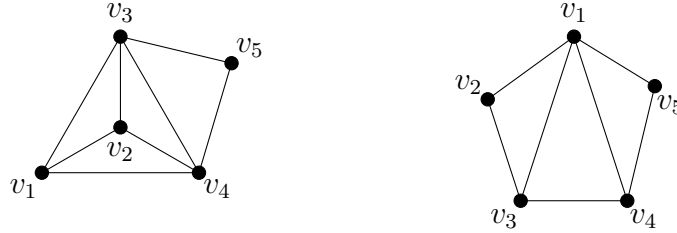


Figure 1: The graph  $\hat{K}_4$  (on the left) and the Gem (on the right).

## 2 Induced minors and well-quasi-ordering

Our main result is the following.

**Theorem 1** (Dichotomy Theorem). *Let  $H$  be a graph. The class of  $H$ -induced minor-free graphs is wqo by  $\leq_{\text{im}}$  iff  $H$  is an induced minor of  $\hat{K}_4$  or the Gem.*

Our proof naturally has two parts: for different values of  $H$ , we need to show wqo of  $H$ -induced minor-free graphs or exhibit an  $H$ -induced minor-free antichain.

**Classes that are wqo.** The following two theorems describe the structure of graphs with  $H$  forbidden as an induced minor, when  $H$  is  $\hat{K}_4$  and the Gem, respectively.

**Theorem 2** (Decomposition of  $\hat{K}_4$ -induced minor-free graphs). *Let  $G$  be a 2-connected graph such that  $\hat{K}_4 \not\leq_{\text{im}} G$ . Then:*

- either  $K_4 \not\leq_{\text{im}} G$ ;
- or  $G$  is a subdivision of a graph among  $K_4$ ,  $K_{3,3}$ , and the prism;
- or  $V(G)$  has a partition  $(C, M)$  such that  $G[C]$  is an induced cycle,  $G[M]$  is a complete multipartite graph and every vertex of  $C$  is either adjacent in  $G$  to all vertices of  $M$ , or to none of them.

**Theorem 3** (Decomposition of Gem-induced minor-free graph). *Let  $G$  be a 2-connected graph such that  $\text{Gem} \not\leq_{\text{im}} G$ . Then  $G$  has a subset  $X \subseteq V(G)$  of at most six vertices such that every connected component of  $G \setminus X$  is either a cograph or a path whose internal vertices are of degree two in  $G$ .*

Using the two above structural results, we are able to show the well-quasi-ordering of the two classes with respect to induced minors. For every graph  $H$ , a graph not containing  $H$  as induced minor is said to be  *$H$ -induced minor-free*.

**Theorem 4.** *The class of  $\hat{K}_4$ -induced minor-free graphs is wqo by  $\leq_{\text{im}}$ .*

**Theorem 5.** *The class of Gem-induced minor-free graphs is wqo by  $\leq_{\text{im}}$ .*

**Organization of the paper.** After a preliminary section introducing notions and notation used in this paper, we present in [Section 4](#) several infinite antichains for induced minors. [Section 5](#) is devoted to the proof of [Theorem 1](#), assuming [Theorem 4](#) and [Theorem 5](#), the proof of which are respectively given in [Section 6](#) and [Section 7](#). Finally, we give in [Section 8](#) some directions for further research.

## 3 Preliminaries

The notation  $\llbracket i, j \rrbracket$  stands for the interval of natural numbers  $\{i, \dots, j\}$ . We denote by  $\mathcal{P}(S)$  the power set of a set  $S$  and by  $\mathcal{P}^{<\omega}(S)$  the set of all its finite subsets.

### 3.1 Graphs and classes

The graphs in this paper are simple and loopless. Given a graph  $G$ ,  $V(G)$  denotes its vertex set and  $E(G)$  its edge set. For every positive integer  $n$ ,  $K_n$  is the complete graph on  $n$  vertices and  $P_n$  is the path on  $n$  vertices. For every integer  $n \geq 3$ ,  $C_n$  is the cycle on  $n$  vertices. For  $H$  and  $G$  graphs, we write  $H + G$  the disjoint union of  $H$  and  $G$ . Also, for every  $k \in \mathbb{N}$ ,  $k \cdot G$  is the disjoint union of  $k$  copies of  $G$ . For every two vertices  $u, v$  of a path  $P$  there is exactly one subpath in  $P$  between  $u$  and  $v$ , that we denote by  $uPv$ . Two vertices  $u, v \in V(G)$  are said to be *adjacent* if  $\{u, v\} \in E(G)$ . The *neighborhood* of  $v \in V(G)$ , denoted  $N_G(v)$ , is the set of vertices that are adjacent to  $v$ . If  $H$  is a subgraph of  $G$ , we write  $N_H(v)$  for  $N_G(v) \cap V(H)$ . Given two sets  $X, Y$  of vertices of a graph, we say that there is an edge between  $X$  and  $Y$  (or that  $X$  and  $Y$  are adjacent) if there is  $x \in X$  and  $y \in Y$  such that  $\{x, y\} \in E(G)$ . The number of connected components of a graph  $G$  is denoted  $\text{cc}(G)$ . We call *prism* the Cartesian product of  $K_3$  and  $K_2$ .

A *cograph* is a graph not containing the path on four vertices as induced subgraph. The following notion will be used when decomposing graphs not containing Gem as induced minor. An induced subgraph of a graph  $G$  is said to be *basic in  $G$*  if it is either a cograph, or an induced path whose internal vertices are of degree two in  $G$ . A *linear forest* is a disjoint union of paths. The *closure* of a class  $\mathcal{G}$  by a given operation is the class obtained from graphs of  $\mathcal{G}$  by a finite application of this operation.

**Complete multipartite graphs.** A graph  $G$  is said to be *complete multipartite* if its vertex set can be partitioned into sets  $V_1, \dots, V_k$  (for some positive integer  $k$ ) in a way such that two vertices of  $G$  are adjacent iff they belong to different  $V_i$ 's. The class of complete multipartite graphs is referred to as  $\mathcal{K}_{\mathbb{N}^*}$ .

**Wheels.** For every positive integer  $k$ , a *k-wheel* is a graph obtained from  $C + K_1$ , where  $C$  is a cycle of order at least  $k$ , by connecting the isolated vertex to  $k$  distinct vertices of the cycle.  $C$  is said to be the *cycle* of the *k-wheel*, whereas the vertex corresponding to  $K_1$  is its *center*.

**Labels and roots.** Let  $(S, \preceq)$  be a poset. A  $(S, \preceq)$ -labeled graph is a pair  $(G, \lambda)$  such that  $G$  is a graph, and  $\lambda: V(G) \rightarrow \mathcal{P}^{<\omega}(S)$  is a function referred as the *labeling of the graph*. For the sake of simplicity, we will refer to the labeled graph of a pair  $(G, \lambda)$  by  $G$  and to  $\lambda$  by  $\lambda_G$ . If  $\mathcal{G}$  is a class of (unlabeled) graphs,  $\text{lab}_{(S, \preceq)}(\mathcal{G})$  denotes the class of  $(S, \preceq)$ -labeled graphs of  $\mathcal{G}$ . Observe that any unlabeled graph can be seen as a  $\emptyset$ -labeled graph. A *rooted graph* is a graph with a distinguished edge called *root*.

Labels will allow us to focus on labelled 2-connected graphs, as stated in the following proposition.

**Proposition 1** ([FHR09]). *Let  $\mathcal{G}$  be a class of graphs. If for any wqo  $(S, \preceq)$  the class of  $(S, \preceq)$ -labeled 2-connected graphs of  $\mathcal{G}$  is wqo by  $\leq_{\text{im}}$ , then  $(\mathcal{G}, \leq_{\text{im}})$  is a wqo.*

## 3.2 Sequences, posets and well-quasi-orders

In this section, we introduce basic definitions and facts related to the theory of well-quasi-orders. In particular, we recall that being well-quasi-ordered is preserved by several operations including union, Cartesian product, and application of a monotone function.

A *sequence* of elements of a set  $A$  is an ordered countable collection of elements of  $A$ . Unless otherwise stated, sequences are finite. The sequence of elements  $s_1, \dots, s_k \in A$  in this order is denoted by  $\langle s_1, \dots, s_k \rangle$ . We use the notation  $A^*$  for the class of all finite sequences over  $A$  (including the empty sequence). The length of a finite sequence  $s \in A^*$  is denoted by  $|s|$ .

A *partially ordered set* (*poset* for short) is a pair  $(A, \preceq)$  where  $A$  is a set and  $\preceq$  is a binary relation on  $S$  which is reflexive, antisymmetric and transitive. An *antichain* is a sequence of pairwise non-comparable elements. In a sequence  $\langle x_i \rangle_{i \in I \subseteq \mathbb{N}}$  of a poset  $(A, \preceq)$ , a pair  $(x_i, x_j)$ ,  $i, j \in I$  is a *good pair* if  $x_i \preceq x_j$  and  $i < j$ . A poset  $(A, \preceq)$  is a

*well-quasi-order* (*wqo* for short)<sup>1</sup>, and its elements are said to be *well-quasi-ordered* by  $\preceq$ , if every infinite sequence has a good pair, or equivalently, if  $(A, \preceq)$  has neither an infinite decreasing sequence, nor an infinite antichain. An infinite sequence containing no good pair is called a *bad sequence*.

A first remark is that every subset  $B$  of a well-quasi-order  $(A, \preceq)$  is well-quasi-ordered by  $\preceq$ . Indeed, any infinite antichain of  $(B, \preceq)$  would also be an antichain of  $(A, \preceq)$ . Let us now look closer at different ways small wqos can be used to build larger ones.

**Union and product.** If  $(A, \preceq_A)$  and  $(B, \preceq_B)$  are two posets, then

- their union  $(A \cup B, \preceq_A \cup \preceq_B)$  is the poset defined as follows:

$$\forall x, y \in A \cup B, x \preceq_A \cup \preceq_B y \text{ if } (x, y \in A \text{ and } x \preceq_A y) \text{ or } (x, y \in B \text{ and } x \preceq_B y);$$

- their Cartesian product  $(A \times B, \preceq_A \times \preceq_B)$  is the poset defined by:

$$\forall (a, b), (a', b') \in A \times B, (a, b) \preceq_A \times \preceq_B (a', b') \text{ if } a \preceq_A a' \text{ and } b \preceq_B b'.$$

*Remark 1* (union of wqos). If  $(A, \preceq_A)$  and  $(B, \preceq_B)$  are two wqos, then so is  $(A \cup B, \preceq_A \cup \preceq_B)$ . In fact, for every infinite antichain  $S$  of  $(A \cup B, \preceq_A \cup \preceq_B)$ , there is an infinite subsequence of  $S$  whose all elements belong to one of  $A$  and  $B$  (otherwise  $S$  is finite). But then one of  $(A, \preceq_A)$  and  $(B, \preceq_B)$  has an infinite antichain, a contradiction with our initial assumption. Similarly, every finite union of wqos is a wqo.

**Proposition 2** (Higman [Hig52]). *If  $(A, \preceq_A)$  and  $(B, \preceq_B)$  are wqo, then so is  $(A \times B, \preceq_A \times \preceq_B)$ .*

**Sequences.** For any partial order  $(A, \preceq)$ , we define the relation  $\preceq^*$  on  $A^*$  as follows: for every  $r = \langle r_1, \dots, r_p \rangle$  and  $s = \langle s_1, \dots, s_q \rangle$  of  $A^*$ , we have  $r \preceq^* s$  if there is an increasing function  $\varphi: \llbracket 1, p \rrbracket \rightarrow \llbracket 1, q \rrbracket$  such that for every  $i \in \llbracket 1, p \rrbracket$  we have  $r_i \preceq s_{\varphi(i)}$ . Observe that  $=^*$  is then the subsequence relation. This order relation is extended to the class  $\mathcal{P}^{<\omega}(A)$  of finite subsets of  $A$  as follows, generalizing the subset relation: for every  $B, C \in \mathcal{P}^{<\omega}(A)$ , we write  $B \preceq^{\mathcal{P}} C$  if there is an injection  $\varphi: B \rightarrow C$  such that  $\forall x \in B, x \preceq \varphi(x)$ . Observe that  $=^{\mathcal{P}}$  is the subset relation.

**Proposition 3** (Higman [Hig52]). *If  $(A, \preceq)$  is a wqo, then so is  $(A^*, \preceq^*)$ .*

**Corollary 1.** *If  $(A, \preceq)$  is a wqo, then so is  $(\mathcal{P}^{<\omega}(A), \preceq^{\mathcal{P}})$ .*

In order to stress that domain and codomain of a function are posets, we sometimes use, in order to denote a function  $\varphi$  from a poset  $(A, \preceq_A)$  to a poset  $(B, \preceq_B)$ , the following notation:  $\varphi: (A, \preceq_A) \rightarrow (B, \preceq_B)$ .

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<sup>1</sup>Usually in literature the term well-quasi-order is defined for more general structures than posets, namely *quasi-orders*. Those relations are like posets, except they are not required to be antisymmetric. This is mere technical detail, as every poset is a quasi-order, and from a quasi-order one can make a poset by taking a quotient by the equivalence relation  $a \preceq b \wedge b \preceq a$ .

**Monotonicity.** A function  $\varphi: (A, \preceq_A) \rightarrow (B, \preceq_B)$  is said to be *monotone* if it satisfies the following property:

$$\forall x, y \in A, x \preceq_A y \Rightarrow f(x) \preceq_B f(y).$$

A function  $\varphi: (A, \preceq_A) \rightarrow (B, \preceq_B)$  is a *poset epimorphism* (*epi* for short) if it is surjective and monotone. Poset epimorphisms have the following interesting property, which we will use to show that some posets are well-quasi-ordered.

*Remark 2* (epi from a wqo). If a domain of epi  $\varphi$  is wqo, then the codomain is also wqo. Indeed, for any pair  $x, y$  of elements of the domain of  $\varphi$  such that  $f(x)$  and  $f(y)$  are incomparable,  $x$  and  $y$  are incomparable as well (by monotonicity of  $\varphi$ ). Therefore, and as  $\varphi$  is surjective, any infinite antichain of the codomain of  $\varphi$  can be translated into an infinite antichain of its domain.

*Remark 3* (componentwise monotonicity). Let  $(A, \preceq_A)$ ,  $(B, \preceq_B)$ , and  $(C, \preceq_C)$  be three posets and let  $f: (A \times B, \preceq_A \times \preceq_B) \rightarrow (C, \preceq_C)$  be a function. If we have both

$$\forall a \in A, \forall b, b' \in B, b \preceq_B b' \Rightarrow f(a, b) \preceq_C f(a, b') \quad (1)$$

$$\text{and } \forall a, a' \in A, \forall b \in B, a \preceq_A a' \Rightarrow f(a, b) \preceq_C f(a', b) \quad (2)$$

then  $f$  is monotone. Indeed, let  $(a, b), (a', b') \in A \times B$  be such that  $(a, b) \preceq_A \times \preceq_B (a', b')$ . By definition of the relation  $\preceq_A \times \preceq_B$ , we have both  $a \preceq_A a'$  and  $b \preceq_B b'$ . From line (1) we get that  $f(a, b) \preceq_C f(a, b')$  and from line (2) that  $f(a, b') \preceq_C f(a', b')$ , hence  $f(a, b) \preceq_C f(a', b')$  by transitivity of  $\preceq_C$ . Thus  $f$  is monotone. Observe that this remark can be generalized to functions with more than two arguments.

### 3.3 Graph operations and containment relations

Most of the common order relations on graphs, sometimes called *containment relations*, can be defined in two equivalent ways: either in terms of *graph operations*, or by using *models*. Let us look closer at them.

**Local operations.** If  $\{u, v\} \in E(G)$ , the *edge contraction* of  $\{u, v\}$  adds a new vertex  $w$  adjacent to the neighbors of  $u$  and  $v$  and then deletes  $u$  and  $v$ . In the case where  $G$  is labeled, we set  $\lambda_G(w) = \lambda_G(u) \cup \lambda_G(v)$ . On the other hand, a *edge subdivision* of  $\{u, v\}$  adds a new vertex adjacent to  $u$  and  $v$  and deletes the edge  $\{u, v\}$ . The *identification* of two vertices  $u$  and  $v$  adds the edge  $\{u, v\}$  if it was not already existing, and contracts it. If  $G$  is  $(\Sigma, \preceq)$ -labeled (for some poset  $(\Sigma, \preceq)$ ), a *label contraction* is the operation of relabeling a vertex  $v \in V(G)$  with a label  $l$  such that  $l \preceq^P \lambda_G(v)$ . The motivation for this definition of label contraction is the following. Most of the time, labels will be used to encode connected graphs into 2-connected graphs. Given a connected graph which is not 2-connected, we can pick an arbitrary block (i.e. a maximal 2-connected component), delete the rest of the graph and label each vertex  $v$  by the subgraph it was attached to in the original graph if  $v$  was a cutvertex, and by  $\emptyset$  otherwise. That way, contracting the label of a vertex  $v$  in the labeled 2-connected graph corresponds to reducing (for some containment relation) the subgraph which was dangling at vertex  $v$  in the original graph.

**Models.** Let  $(\Sigma, \preceq)$  be any poset. A *containment model* of a  $(\Sigma, \preceq)$ -labeled graph  $H$  in a  $(\Sigma, \preceq)$ -labeled graph  $G$  ( $H$ -model for short) is a function  $\mu: V(H) \rightarrow \mathcal{P}^{<\omega}(V(G))$  satisfying the following conditions:

- (i) for every two distinct  $u, v \in V(H)$ , the sets  $\mu(u)$  and  $\mu(v)$  are disjoint;
- (ii) for every  $u \in V(H)$ , the subgraph of  $G$  induced by  $\mu(u)$  is connected;
- (iii) for every  $u \in V(H)$ ,  $\lambda_H(u) \preceq^* \bigcup_{v \in \mu(u)} \lambda_G(v)$  (label conservation).

When in addition  $\mu$  is such that for every two distinct  $u, v \in V(H)$ , the sets  $\mu(u)$  and  $\mu(v)$  are adjacent in  $G$  iff  $\{u, v\} \in E(H)$ , then  $\mu$  is said to be an *induced minor model* of  $H$  in  $G$ .

If  $\mu$  is an induced minor model of  $H$  in  $G$  satisfying the following condition:

$$\bigcup_{v \in V(H)} \mu(v) = V(G),$$

then  $\mu$  is a *contraction model* of  $H$  in  $G$ .

If  $\mu$  is an induced minor model of  $H$  in  $G$  satisfying the following condition:

$$\forall v \in V(H), |\mu(v)| = 1,$$

then  $\mu$  is an *induced subgraph model* of  $H$  in  $G$ .

An  $H$ -model in a graph  $G$  witnesses the presence of  $H$  as substructure of  $G$  (which can be induced subgraph, induced minor, contraction, etc.), and the subsets of  $V(G)$  given by the image of the model indicate which subgraphs to keep and to contract in  $G$  in order to obtain  $H$ .

When dealing with rooted graphs, the aforementioned models must in addition preserve the root, that is, if  $\{u, v\}$  is the root of  $H$  then the root of  $G$  must have one endpoint in  $\mu(u)$  and the other in  $\mu(v)$ .

**Containment relations.** Local operations and models can be used to express that a graph is *contained* in an other one, for various definitions of “contained”. We say that a graph  $H$  is an *induced minor* (resp. a contraction, induced subgraph) of a graph  $G$  if there is an induced minor model (resp. a contraction model, an induced subgraph model)  $\mu$  of  $H$  in  $G$ , what we note  $H \leq_{\text{im}}^\mu G$  (resp.  $H \leq_c^\mu G$ ,  $H \leq_{\text{isg}}^\mu G$ ), or simply  $H \leq_{\text{im}} G$  (resp.  $H \leq_c G$ ,  $H \leq_{\text{isg}} G$ ) when the model is not specified.

Otherwise,  $G$  is said to be  *$H$ -induced minor-free* (resp.  *$H$ -contraction-free*,  *$H$ -induced subgraph-free*). The class of  $H$ -induced minor-free graphs will be referred to as  $\text{Excl}_{\text{im}}(H)$ .

*Remark 4.* In terms of local operation, these containment relations are defined as follows for every  $H, G$  graphs:

- $H \leq_{\text{isg}} G$  iff there is a (possibly empty) sequence of vertex deletions and label contractions transforming  $G$  into  $H$ ;
- $H \leq_{\text{im}} G$  iff there is a (possibly empty) sequence of vertex deletions, edge contractions and label contractions transforming  $G$  into  $H$ ;
- $H \leq_c G$  iff there is a (possibly empty) sequence of edge contractions and label contractions transforming  $G$  into  $H$ .

**Subdivisions.** A subdivision of a graph  $H$ , or  $H$ -subdivision, is a graph obtained from  $H$  by edge subdivisions. The vertices added during this process are called *subdivision vertices*.

**Containing  $K_4$ -subdivisions.** A graph  $G$  contains  $K_4$  as an induced minor if and only if  $G$  contains  $K_4$ -subdivision as a subgraph. This equivalence is highly specific to the graph  $K_4$  and in general neither implication would be true. We will freely change between those two notions for containing  $K_4$ , depending on which one is more convenient in the given context.

A graph  $G$  will be said to contain a *proper  $K_4$  subdivision*, if there is some vertex  $v \in V(G)$ , such that  $G \setminus v$  contains a  $K_4$ -subdivision.

### 3.4 Structure and decompositions

**Cycle-multipartite.** Given a graph  $G$ , a pair  $(C, R)$  of induced subgraphs of  $G$  is said to be a *cycle-multipartite decomposition* of  $G$  if the following conditions are satisfied:

- (i)  $(V(C), V(R))$  is a partition of  $V(G)$ ;
- (ii)  $C$  is a cycle and  $R$  is a complete multipartite graph;
- (iii)  $\forall u, v \in V(R), N_C(u) = N_C(v)$ .

The class of graphs having cycle-multipartite decomposition is denoted by  $\mathcal{W}$ .

**Cuts.** In a graph  $G$ , a  $K_2$ -cut (resp.  $\overline{K_2}$ -cut) is a subset  $S \subseteq V(G)$  such that  $G - S$  is not connected and  $G[S]$  is isomorphic to  $K_2$  (resp.  $\overline{K_2}$ ).

## 4 Antichains for induced minors

An infinite antichain is an obstruction for a quasi-order to be a wqo. As we will see in [Section 5](#), the study of infinite antichains can provide helpful information when looking for graphs  $H$  such that  $(\text{Excl}_{\text{im}}(H), \leq_{\text{im}})$  is a wqo. Let us enumerate some of the known infinite antichains for induced minors.

In 1985, Thomas [[Tho85](#)] presented an infinite sequence of planar graphs (also mentioned later in [[RS93](#)]) and proved that it is an antichain for induced minors. He showed that this relation does not well-quasi-order planar graphs. The elements of this antichain, called *alternating double wheels*, are constructed from an even cycle by adding two nonadjacent vertices and connecting one to one color class of the cycle, and connecting the other vertex to the other color class (cf. [Figure 2](#) for the three first such graphs). This infinite antichain shows that  $(\text{Excl}_{\text{im}}(K_5), \leq_{\text{im}})$  is not a wqo since no alternating double wheel contains  $K_5$  as (induced) minor. As a consequence,  $(\text{Excl}_{\text{im}}(H), \leq_{\text{im}})$  is not a wqo as soon as  $H$  contains  $K_5$  as induced minor.

Therefore, in the quest for all graphs  $H$  such that  $(\text{Excl}_{\text{im}}(H), \leq_{\text{im}})$  is wqo, we can focus the cases where  $H$  is  $K_5$ -induced minor-free.



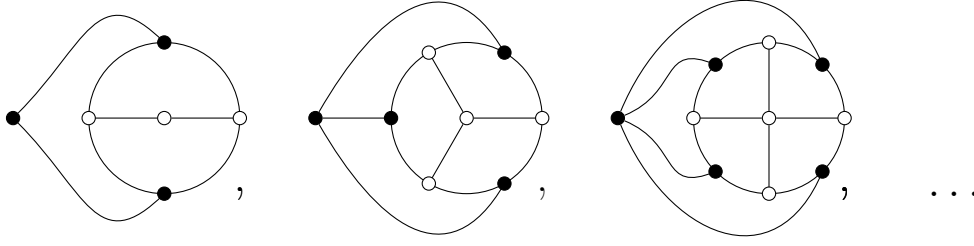


Figure 2: Thomas' alternating double wheels.

The infinite antichain  $\mathcal{A}_M$  depicted in [Figure 3](#) was introduced in [\[MNT88\]](#), where it is also proved that none of its members contains  $K_5^-$  as induced minor. Similarly as the above remark, it follows that if  $(\text{Excl}_{\text{im}}(H), \leq_{\text{im}})$  is a wqo then  $K_5^- \not\leq_{\text{im}} H$ . Notice that graphs in this antichain have bounded maximum degree.

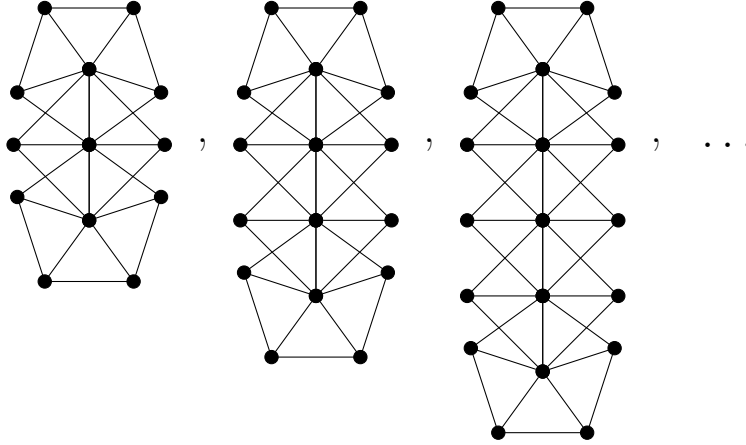


Figure 3: The infinite antichain  $\mathcal{A}_M$

An *interval graph* is the intersection graph of segments of  $\mathbb{R}$ . A well-known property of interval graphs that we will use later is that they do not contain  $C_4$  as induced minor. In order to show that interval graphs are not wqo by  $\leq_{\text{im}}$ , Ding introduced in [\[Din98\]](#) an infinite sequence of graphs defined as follows. For every  $n \in \mathbb{N}$ ,  $n > 2$ , let  $T_n$  be the set of closed intervals

- $[i, i]$  for  $i$  in  $\llbracket -2n, -1 \rrbracket \cup \llbracket 1, 2n \rrbracket$ ;
- $[-2, 2]$ ,  $[-4, 1]$ ,  $[-2n + 1, 2n]$ ,  $[-2n + 1, 2n - 1]$ ;
- $[-2i + 1, 2i + 1]$  for  $i$  in  $\llbracket 1, n - 2 \rrbracket$ ;
- $[-2i, 2i - 2]$  for  $i$  in  $\llbracket 3, n \rrbracket$ .

[Figure 4](#) depicts the intervals of  $T_6$ : the real axis (solid line) is folded up and an interval  $[a, b]$  is represented by a dashed line between  $a$  and  $b$ .

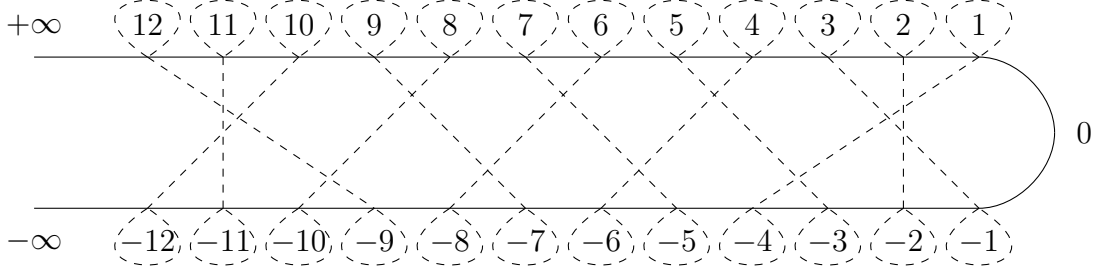


Figure 4: An illustration of the intervals in  $T_6$ .

For every  $n \in \mathbb{N}$ ,  $n > 2$ , let  $A_n^D$  be the intersection graph of segments of  $T_n$ . Let  $\mathcal{A}_D = \langle A_n^D \rangle_{n \geq 2}$ . Ding proved in [Din98] that  $\mathcal{A}_D$  is an antichain for  $\leq_{\text{im}}$ , thus showing that interval graphs are not wqo by induced minors.

Let us now present two infinite antichains that were, to our knowledge, not mentioned elsewhere earlier. Let  $\mathcal{A}_{\overline{C}} = \langle \overline{C}_n \rangle_{n \geq 6}$  be the sequence of *antiholes* of order at least six, whose first elements are represented in Figure 5.

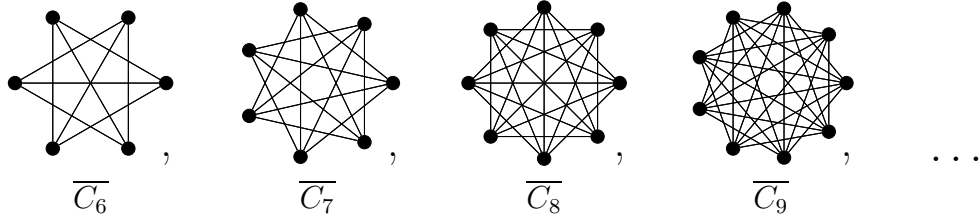


Figure 5: Antiholes antichain.

**Lemma 1.**  $\mathcal{A}_{\overline{C}}$  is an antichain.

*Proof.* Observe that each vertex deletion correspond to a vertex deletion in the complement graph, and each edge contraction corresponds to an vertex identification of its endpoints in the complement graph. Notice that performing any of these operations on a cycle would yield a linear forest, and performing any of those operations on a linear forest, yields a linear forest again.

Therefore, it is not possible to obtain  $C_k$  from  $C_n$ , where  $k < n$ , by a sequence of vertex deletions and vertex identifications of non-adjacent vertices. Equivalently, it is not possible to obtain  $\overline{C}_k$  from  $\overline{C}_n$  by a sequence of vertex deletions and contractions. This proves that  $\mathcal{A}_{\overline{C}}$  is an antichain with respect to  $\leq_{\text{im}}$ .  $\square$

We will meet again the antichain  $\mathcal{A}_{\overline{C}}$  in the proof of Theorem 1. Another infinite antichain which shares with  $\mathcal{A}_M$  the properties of planarity and bounded maximum degree is depicted in Figure 6. We will not go more into detail about it here as this antichain will be of no use in the rest of the paper.

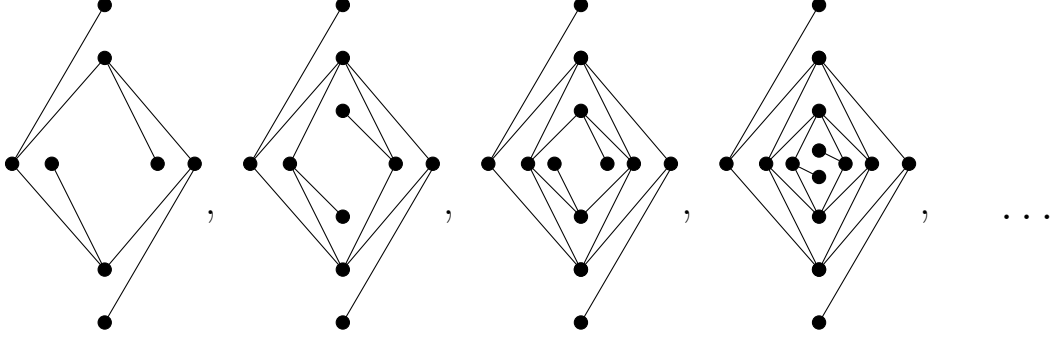


Figure 6: Nested lozenges.

## 5 The dichotomy theorem

The purpose of this section is to prove [Theorem 1](#), that is, to characterize all graphs  $H$  such that  $(\text{Excl}_{\text{im}}(H), \leq_{\text{im}})$  is a wqo. To this end, we will assume [Theorem 4](#) and [Theorem 5](#), which we will prove later, in [Section 6](#) and [Section 7](#) respectively.

The main ingredients of the proof are the infinite antichains presented in [Section 4](#), together with [Theorem 4](#) and [Theorem 5](#). Infinite antichains will be used to discard every graph  $H$  that is not induced minor of all but finitely many elements of some infinite antichain. On the other hand, knowing that  $(\text{Excl}_{\text{im}}(H), \leq_{\text{im}})$  is a wqo gives that  $(\text{Excl}_{\text{im}}(H'), \leq_{\text{im}})$  is a wqo for every  $H' \leq_{\text{im}} H$  in the virtue of the following remark.

*Remark 5.* For every  $H, H'$  such that  $H' \leq_{\text{im}} H$ , we have  $\text{Excl}_{\text{im}}(H') \subseteq \text{Excl}_{\text{im}}(H)$ .

Let  $H$  be any graph.

**Lemma 2.** *If  $(\text{Excl}_{\text{im}}(H), \leq_{\text{im}})$  is a wqo then  $\overline{H}$  is a linear forest.*

*Proof.* Let us show that  $\text{Excl}_{\text{im}}(H)$  has an infinite antichain given that  $\overline{H}$  is not a linear forest. In this case,  $\overline{H}$  either has a vertex of degree at least 3 or it contains an induced cycle as induced subgraph.

*First case:*  $\overline{H}$  has a vertex  $v$  of degree 3. Let  $x, y, z$  be three neighbors of  $v$ . In the graph  $H[\{v, x, y, z\}]$ , the vertex  $v$  is adjacent to none of  $x, y, z$ . In an antihole, every vertex has exactly two non-neighbors, so  $H[\{v, x, y, z\}]$  is not an induced minor of any element of  $\mathcal{A}_{\overline{C}}$ . Therefore  $\mathcal{A}_{\overline{C}} \subseteq \text{Excl}_{\text{im}}(H)$ .

*Second case:*  $\overline{H}$  contains an induced cycle as an induced subgraph. Let us first assume that for some integer  $k \geq 6$  we have  $\overline{C}_k \leq_{\text{im}} H$ . Now consider any  $\overline{C}_n$  for  $n > |H|$ . Clearly, we have  $\overline{C}_n \not\leq_{\text{im}} H$ . On the other hand if  $H \leq_{\text{im}} \overline{C}_n$ , then by the fact that  $\overline{C}_k \leq_{\text{im}} H$  and transitivity, we would have  $\overline{C}_k \leq_{\text{im}} \overline{C}_n$ , which would yield a contradiction with the fact that  $\mathcal{A}_{\overline{C}}$  is an antichain. Hence  $\mathcal{A}_{\overline{C}} \cap \text{Excl}_{\text{im}}(H)$  contains all antiholes of size greater than  $|H|$ , and in particular is infinite as required. In the cases where  $\overline{H}$  has a cycle of length 3, 4 or 5, it is easy to check that no element of  $\mathcal{A}_{\overline{C}}$  contains (respectively) three independent vertices, two independent edges, or an edge not adjacent to an other vertex (which is an induced subgraph of  $\overline{C}_5$ ).  $\square$

Due to the interesting properties on  $\overline{H}$  given by [Lemma 2](#), we will be led below to work with this graph rather than with  $H$ . The following lemma presents step by step the properties that we can deduce on  $\overline{H}$  by assuming that  $\text{Excl}_{\text{im}}(H)$  is wqo by  $\leq_{\text{im}}$ .

**Lemma 3.** *If  $(\text{Excl}_{\text{im}}(H), \leq_{\text{im}})$  is a wqo, then we have*

- (R1)  $\overline{H}$  has at most 4 connected components;
- (R2) at most one connected component of  $\overline{H}$  is not a single vertex;
- (R3) the largest connected component of  $\overline{H}$  has at most 4 vertices;
- (R4) if  $n = |V(H)|$  and  $c = \text{cc}(\overline{H})$  then  $n \leq 7$  and  $\overline{H} = (c - 1) \cdot K_1 + P_{n-c+1}$ ;
- (R5) if  $\text{cc}(\overline{H}) = 3$  then  $|V(H)| \leq 5$ .
- (R6) if  $\text{cc}(\overline{H}) = 4$  then  $|V(H)| \leq 4$ .

*Proof.* *Proof of item (R1).* The infinite antichain  $\mathcal{A}_M$  does not contain  $K_5$  and (induced) minor, hence  $K_5 \not\leq_{\text{im}} H$  and so  $\overline{H}$  does not contain  $5 \cdot K_1$  as induced minor. Therefore it has at most 4 connected components.

*Proof of items (R2) and (R3).* The infinite antichain  $\mathcal{A}_D$  does not contain  $C_4$  as induced minor (as it is an interval graph), hence neither does  $H$ . Therefore  $\overline{H}$  does not contain  $2 \cdot P_2$  as induced minor. This implies that  $\overline{H}$  does not contain  $P_5$  as induced minor and that given two connected components of  $\overline{H}$  at least one must be of order one. As connected components of  $H$  are paths (by [Lemma 2](#)), the largest connected component of  $H$  has order at most 4.

Item (R4) follows from the above proofs and from the fact that  $\overline{H}$  is a linear forest. *Proof of item (R5).* Similarly as in the proof of item (R1),  $\mathcal{A}_M$  does not contain  $K_5^-$  as induced minor so  $\overline{K_5^-} = K_2 + 3 \cdot K_1$  is not an induced minor of  $\overline{H}$ . If we assume that  $\text{cc}(\overline{H}) = 3$  and  $|V(H)| \geq 6$  vertices, the largest component of  $\overline{H}$  is a path on (a least) 4 vertices, so it contains  $K_1 + K_2$  as induced subgraph. Together with the two other (single vertex) components, this gives an  $K_2 + 3 \cdot K_1$  induced minor, a contradiction. *Proof of item (R6).* Let us assume that  $\text{cc}(\overline{H}) = 4$ . If the largest connected component has more than one vertex, then  $\overline{H}$  contains  $K_2 + 3 \cdot K_1$  induced minor, which is not possible (as in the proof of item (R5)). Therefore  $\overline{H} = 4 \cdot K_1$  and so  $|V(\overline{H})| = 4$ .  $\square$

We are now able to describe more precisely graphs  $H$  for which  $(\text{Excl}_{\text{im}}(H), \leq_{\text{im}})$  could be a wqo. Let  $K_3^+$  be the complement of  $P_3 + K_1$  and let  $K_4^-$  be the complement of  $K_2 + 2 \cdot K_1$ , which is also the graph obtained from  $K_4$  by deleting an edge (sometimes referred as *diamond graph*).

**Lemma 4.** *If  $(\text{Excl}_{\text{im}}(H), \leq_{\text{im}})$  is a wqo, then  $H \leq_{\text{im}} \hat{K}_4$  or  $H \leq_{\text{im}} \text{Gem}$ .*

*Proof.* Using the information on  $\overline{H}$  given by [Lemma 3](#), we can build a table of possible graphs  $\overline{H}$  depending on  $\text{cc}(\overline{H})$  and  $|V(\overline{H})|$ . [Table 1](#) is such a table: each column corresponds for a number of connected components (between one and four according to item (R1)) and each line corresponds to an order (at most seven, by item (R4)). A grey cell means either that there is no such graph (for instance a graph with one vertex and two connected components), or that for all graphs  $\overline{H}$  matching the number of connected

$ V(H)  \setminus \text{cc}(\overline{H})$	1	2	3	4
1	$K_1$			
2	$K_2$	$2 \cdot K_1$		
3	$P_3$	$K_2 + K_1$	$3 \cdot K_1$	
4	$P_4$	$P_3 + K_1$	$K_2 + 2 \cdot K_1$	$4 \cdot K_1$
5	(R3)	$P_4 + K_1$	$P_3 + 2 \cdot K_1$	(R6)
6	(R3)	(R3)	(R5)	(R6)
7	(R3)	(R3)	(R5)	(R6)

Table 1: If  $(\text{Excl}_{\text{im}}(H), \leq_{\text{im}})$  is a wqo, then  $\overline{H}$  belongs to this table.

$ V(H)  \setminus \text{cc}(\overline{H})$	1	2	3	4
1	$K_1$			
2	$2 \cdot K_1$	$K_2$		
3	$K_2 + K_1$	$P_3$	$K_3$	
4	$P_4$	$K_3^+$	$K_4^-$	$K_4$
5	(R3)	Gem	$\hat{K}_4$	(R6)
6	(R3)	(R3)	(R5)	(R6)
7	(R3)	(R3)	(R5)	(R6)

Table 2: If  $(\text{Excl}_{\text{im}}(H), \leq_{\text{im}})$  is a wqo, then  $H$  belongs to this table.

components and the order associated with this cell, the poset  $(\text{Excl}_{\text{im}}(H), \leq_{\text{im}})$  is not a wqo.

From Table 1 we can easily deduce Table 2 of corresponding graphs  $H$ .

Remark that we have

- $K_1 \leq_{\text{im}} 2 \cdot K_1 \leq_{\text{im}} K_2 + K_1 \leq_{\text{im}} P_4 \leq_{\text{im}} \text{Gem}$ ;
- $K_2 \leq_{\text{im}} P_3 \leq_{\text{im}} K_3^+ \leq_{\text{im}} \text{Gem}$ ;
- $K_3 \leq_{\text{im}} K_4^- \leq_{\text{im}} \hat{K}_4$ ; and
- $K_4 \leq_{\text{im}} \hat{K}_4$ .

This concludes the proof. □

We are now ready to give the proof of Theorem 1.

*Proof of Theorem 1.* If  $H \not\leq_{\text{im}} \text{Gem}$  and  $H \not\leq_{\text{im}} \hat{K}_4$ , then by Lemma 4  $(\text{Excl}_{\text{im}}(H), \leq_{\text{im}})$  is not a wqo. On the other hand, by Theorem 4 and Theorem 5 we know that both  $\text{Excl}_{\text{im}}(\hat{K}_4)$  and  $\text{Excl}_{\text{im}}(\text{Gem})$  are wqo by  $\leq_{\text{im}}$ . Consequently, by Remark 5,  $(\text{Excl}_{\text{im}}(H), \leq_{\text{im}})$  is wqo as soon as  $H \leq_{\text{im}} \text{Gem}$  or  $H \leq_{\text{im}} \hat{K}_4$ . □

## 6 Graphs not containing $\hat{K}_4$

The main goal of this section is to provide a proof to [Theorem 4](#). To this purpose, we first prove in [Subsection 6.1](#) that graphs of  $\text{Excl}_{\text{im}}(\hat{K}_4)$  admits a simple structural decomposition. This structure is then used in [Subsection 6.2](#) to show that graphs of  $\text{Excl}_{\text{im}}(\hat{K}_4)$  are well-quasi-ordered by the relation  $\leq_{\text{im}}$ .

### 6.1 A decomposition theorem for $\text{Excl}_{\text{im}}(\hat{K}_4)$

The main topic of this section is the proof of [Theorem 2](#). This theorem states that every graph in the class  $\text{Excl}_{\text{im}}(\hat{K}_4)$ , either does not have even  $K_4$  as induced minor, or is a subdivision of some small graph, or has a cycle-multipartite decomposition. Most of the time, we show that some property  $P$  is not satisfied by graphs of  $\text{Excl}_{\text{im}}(\hat{K}_4)$  by showing an induced minor model of  $\hat{K}_4$  in graphs satisfying  $P$ . We first assume that  $G$  contains a proper  $K_4$ -subdivision, and we show in [Lemma 14](#) how to deal with the other case.

**Lemma 5.** *If  $G$  contains as induced minor any graph  $H$  consisting of:*

- *a  $K_4$ -subdivision  $S$ ;*
- *an extra vertex  $x$  linked by exactly two paths  $L_1$  and  $L_2$  to two distinct vertices  $s_1, s_2 \in V(S)$ , where the only common vertex of  $L_1$  and  $L_2$  is  $x$ ;*
- *and possibly extra edges between the vertices of  $S$ , or between  $L_1$  and  $L_2$ , or between the interior of the paths and  $S$ ,*

*then  $\hat{K}_4 \leq_{\text{im}} G$ .*

*Proof.* Let us call  $V = \{v_1, v_2, v_3, v_4\}$  the non-subdivision vertices of  $S$ , i.e. vertices corresponding to vertices of  $K_4$ . We present here a sequence of edge contractions in  $H$  leading to  $\hat{K}_4$ . Let us repeat the following procedure: as long as there is a path between two elements of  $V \cup \{s_1, s_2, s\}$ , internally disjoint with this set, contract the whole path to a single edge.

Once we can not apply this contraction any more, we end up with a graph that has two parts: the  $K_4$ -subdivision with at most 2 subdivisions (with vertex set  $V \cup \{s_1, s_2\}$ ) and the vertex  $x$ , which is now only adjacent to  $s_1$  and  $s_2$ .

*First case:*  $s_1, s_2 \in V$ . The graph  $H$  is isomorphic to  $\hat{K}_4$ : it is  $K_4$  plus a vertex of degree two.

*Second case:*  $s_1 \in V$  and  $s_2 \notin V$  (and the symmetric case). As vertices of  $V$  are the only vertices of  $H$  that have degree 3 in  $S$ ,  $s_2$  is of degree 2 in  $S$  (it is introduced by subdivision). The contraction of the edge between  $s_2$  and one of its neighbors in  $S$  that is different to  $s_1$  leads to first case.

*Third case:*  $s_1, s_2 \notin V$ . As in second case, these two vertices have degree two in  $S$ . Since no two different edges of  $K_4$  can have the same endpoints, the neighborhoods of  $s_1$  and  $s_2$  have at most one common vertex. Then for every  $i \in \{1, 2\}$  there is a neighbor  $t_i$  of  $s_i$  that is not adjacent to  $s_{3-i}$ . Contracting the edges  $\{s_1, t_1\}$  and  $\{s_2, t_2\}$  leads to first case.  $\square$

**Corollary 2** (Proof of [Lemma 5](#)). *Let  $G \in \text{Excl}_{\text{im}}(\hat{K}_4)$  be a biconnected graph containing a proper  $K_4$ -subdivision. For every subdivision  $S$  of  $K_4$  in  $G$ , and for every vertex  $x \in V(G) \setminus V(S)$ ,  $N_S(x) \geq 3$ .*

*Proof.* Let  $S$  be a proper  $K_4$ -subdivision in  $G$  and  $x \in V(G) \setminus V(S)$ . Let  $L_1, L_2$  be two shortest paths from  $x$  to  $S$  meeting only in  $x$ . Such paths exist by the biconnectivity of  $G$ . Remark that if  $|N_G(x) \cap V(S)| \leq 2$ , then the graph induced by  $S$ ,  $L_1$ , and  $L_2$  satisfies conditions of [Lemma 5](#). Therefore,  $N_S(x) \geq 3$ .  $\square$

*Remark 6.* For every two edges of  $K_4$  there is a Hamiltonian cycle using these edges.

*Remark 7.* Three edges of  $K_4$  are not contained into a same cycle iff they are incident with the same vertex.

**Lemma 6.** *Every biconnected graph  $G \in \text{Excl}_{\text{im}}(\hat{K}_4)$  containing a proper  $K_4$ -subdivision has a 3-wheel as subgraph.*

*Proof.* Let  $S$  be a minimum (proper)  $K_4$ -subdivision in  $G$  and let  $x \in V(G) \setminus V(S)$ . We define  $V$  as in the proof of [Lemma 5](#) and we say that two neighbours of  $x$  in  $S$  are equivalent if they lie on the same path between two elements of  $V$ , (intuitively they correspond to the same edge of  $K_4$ ). By [Corollary 2](#), we only have to consider the case  $|N_S(x)| \geq 3$ .

First of all, remark that if some three neighbors of  $x$  lie on a cycle of  $S$ , then we are done. Let us assume from now on, that there is no cycle of  $S$  containing three neighbors of  $x$ . This implies that no two neighbors of  $x$  are equivalent (by [Remark 6](#)), no neighbor of  $x$  belongs to  $V$  (by the same remark) and that  $|N_S(x)| = 3$  (by [Remark 7](#)). Let us consider the induced minor  $H$  of  $S + x$  obtained by contracting all edges not incident with two vertices of  $V \cup N_S(x)$ . By [Remark 7](#) and since the three neighbors of  $x$  do not belong to a cycle, there is a vertex of  $V(H) \setminus \{x\}$  adjacent to the three neighbors of  $x$ . Contracting two of the edges incident with this vertex merges two neighbors of  $x$  and the graph we obtain is a  $K_4$  subdivision (corresponding to  $S$ ) together with a vertex of degree 2 (corresponding to  $x$ ). By [Lemma 5](#), this would imply that  $\hat{K}_4 \leq_{\text{im}} G$ , a contradiction. Therefore three neighbors of  $x$  lie on a cycle of  $S$  and this concludes the proof.  $\square$

Now we will deal with a graph  $G$  that satisfies conditions of [Lemma 6](#); namely:  $G$  is a biconnected graph, without  $\hat{K}_4$  as an induced minor, containing a proper  $K_4$ -subdivision;  $C$  denotes the cycle of a minimum (in terms of number of vertices) 3-wheel in  $G$ , and  $R$  the graph induced by the remaining vertices.

*Remark 8.* As this 3-wheel is a subdivision of  $K_4$  as subgraph of  $G$ , by [Corollary 2](#), every vertex of  $R$  has at least three neighbors in  $C$ .

*Remark 9.* This 3-wheel contains no more vertices than the  $K_4$ -subdivision that we assumed to be contained in  $G$ . Therefore, every minimum  $K_4$ -subdivision of  $G$  (in terms of vertices) is a 3-wheel.

**Lemma 7.** *Let  $G$  be a biconnected graph of  $\text{Excl}_{\text{im}}(\hat{K}_4)$ . Every minimum (in terms of number of vertices) 3-wheel  $W$  of cycle  $C$  and center  $r$  that is a subgraph of  $G$  is such that, if  $C$  is not an induced cycle in  $G$ ,*

- (i) the endpoints of every chord are both adjacent to some  $u \in N_C(r)$ ;
- (ii) every two distinct  $v, w \in N_C(r) \setminus \{u\}$  are adjacent on  $C$ ;
- (iii)  $C$  has exactly one chord in  $G$
- (iv)  $|N_C(r)| = 3$ .

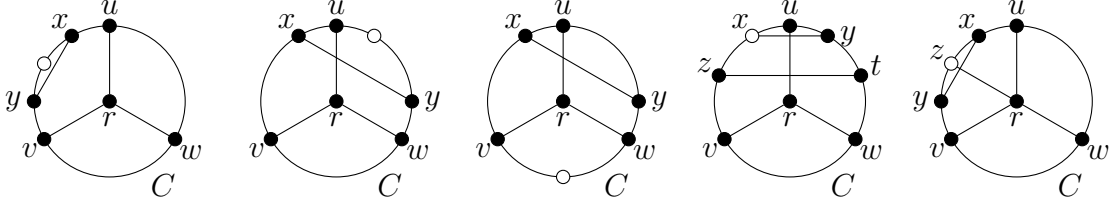


Figure 7: Forbidden configurations in the proof of Lemma 7.

*Proof.* Let  $u, v, w \in N_C(r)$  be three distinct neighbours of  $r$  in  $C$  and let  $C_u$  be the path of  $C$  between  $v$  and  $w$  that does not contain the vertex  $u$ , and similarly for  $C_v$  and  $C_w$ . First of all, notice that no proper subgraph of  $W$  can be a subdivision of  $K_4$ , otherwise  $G$  would contain a graph smaller than  $W$  but meeting the same requirements, according to Lemma 6. Below we will show that when conditions (i)-(iv) are not fulfilled,  $W$  is not minimal, i.e. that when some vertices are deleted in  $W$ , it still contains a  $K_4$ -subdivision. Figure 7 illustrates such configurations, where white vertices can be deleted.

Let  $\{x, y\}$  be a chord of  $C$  in  $G$ . Remark that the endpoints of a chord cannot belong to the same path  $C_l$  for any  $l \in \{u, v, w\}$  without violating the minimality of  $S$ , as deleting vertices of  $C_l$  that are between  $x$  and  $y$  would still lead to a 3-wheel (first configuration of Figure 7). Therefore,  $x$  and  $y$  belongs to different  $C_l$ 's, say without loss of generality that  $x \in V(C_w)$  and  $y \in V(C_v)$ .

Let us prove that  $x$  and  $y$  must both be adjacent to  $u$ . By contradiction, we assume that, say,  $y$  and  $u$  are not adjacent. Let us consider the induced subgraph of  $W$  obtained by the deletion of the interior of the path  $yC_vu$  (containing at least one vertex). Notice that contracting each of the paths  $vC_uw$ ,  $vC_wx$ ,  $yC_vw$  and  $(r, u) - uC_wx$  gives  $K_4$ , a contradiction (cf. the second configuration of Figure 7). The case where  $x$  and  $u$  are not adjacent is symmetric. This proves that every chord of  $C$  in  $G$  has endpoints adjacent to a same neighbor of  $r$  on  $C$ , that is (i).

Now, we show that the path  $C_u$  must be an edge. To see this, assume by contradiction that it has length at least 3. The subgraph of  $W$  induced by the six paths  $(r, v) - vC_wx$ ,  $\{r, u\}$ ,  $(r, w) - wC_vy$ ,  $\{x, y\}$ ,  $uC_wx$  and  $uC_vy$  does not contain the internal vertices of  $C_u$ , thus it is smaller than  $W$  (third configuration of Figure 7). However, it contains a subdivision of  $K_4$  as subgraph, that can for instance be obtained by contracting each of these six paths to an edge. This is contradictory according to our first remark, therefore the two neighbors  $v$  and  $w$  of  $r$  on the cycle are adjacent. This proves item (ii).

Let now assume that there  $C$  has a second chord  $\{z, t\}$  in  $G$ . In the light of the previous remark,  $C_u$  is an edge, hence the only paths to which  $z$  and  $t$  can belong are



the paths  $xC_wv$  and  $yC_vw$  and, according to our first remark, they do not both belong to the same of these two paths. Also, as  $\{z, t\} \neq \{x, y\}$ , one of  $z, t$  does not belong to  $\{x, y\}$ . We can thus assume without loss of generality that  $z \in V(xC_wv)$ ,  $t \in V(yC_vw)$  and  $z \neq x$ . This case is represented by the fourth configuration of [Figure 7](#). Let us consider the cycle  $C'$  obtained by the concatenation of the paths  $zC_wv$ ,  $C_u$ ,  $wC_vt$  and  $(t, z)$  and the vertex  $r$ , which is connected to the cycle by the three paths  $(r, v)$ ,  $(r, w)$  and  $(r, u) - uC_vt$  that only share the vertex  $r$ . This subgraph is smaller than  $H$  since it does not contain vertex  $x$ , but it is a subdivision of  $K_4$  (three paths issued from the same vertex  $a$  and meeting the cycle  $C'$ ). By one of the above remarks, this configuration is impossible and thus the chord  $(z, t)$  cannot exist. Hence we proved item (iii):  $C$  can have at most one chord in  $G$ .

Remark that we have  $|N_C(r)| \geq 3$  since  $W$  is a 3-wheel. We now assume that  $|N_C(r)| > 3$ . Let  $u, v, w, z$  be four different neighbors of  $r$  such that  $z$  is a common neighbor of the endpoints of the chord  $x$  and  $y$ . This case is depicted in the fifth configuration of [Figure 7](#). Then  $r$  has at least three neighbors ( $u, v$ , and  $w$ ) on the cycle going through the edge  $\{x, y\}$  and following  $C$  up to  $x$  without using the vertex  $z$ . As this contradicts the minimality of  $W$ , we have  $|N_C(r)| = 3$ , that is item (iv), and this concludes the proof.  $\square$

**Corollary 3.** *According to [Remark 9](#), every minimum  $K_4$ -subdivision in  $G$  is a 3-wheel, so [Lemma 7](#) is still true when replacing 3-wheel by  $K_4$ -subdivision in its statement.*

**Lemma 8.** *Every two non-adjacent vertices of  $R$  have the same neighborhood in  $C$ .*

*Proof.* By contradiction, we assume that there are two non-adjacent vertices  $s, t \in V(R)$  and a vertex  $u_1 \in V(C)$  such that  $\{s, u_1\} \in E(G)$  but  $\{t, u_1\} \notin E(G)$ . By [Remark 8](#),  $s$  and  $t$  have (at least) three neighbors in  $C$ . Let  $U = \{u_1, u_2, u_3\}$  and  $V = \{v_1, v_2, v_3\}$  be the respective neighbors of  $s$  and  $t$ . We consider the graph  $H$  induced in  $G$  by  $C$  and  $\{s, t\}$  where we iteratively contracted every edge of  $S$  not incident with to two vertices of  $\{u_1, u_2, u_3, v_1, v_2, v_3\}$ . This graph is a (non necessarily induced) cycle on at most 6 vertices, that we call  $C'$  plus the two non-adjacent vertices of degree at least three  $s$  and  $t$ . Remark that while two neighbors of  $s$  are adjacent and are not both neighbors of  $t$ , we can contract the edge between them and decrease the degree of  $s$ , without changing degree of  $t$ . If the degree of  $s$  reaches two by such means, then by [Lemma 5](#),  $\hat{K}_4 \leq_{\text{im}} H$ , a contradiction. We can thus assume that every vertex of  $C'$  adjacent to a neighbor of  $s$  is a neighbor of  $t$ . This is also true when  $s$  and  $t$  are swapped since this argument can be applied to  $t$  too. This observation implies that  $N_S(s) \cap N_S(t) = \emptyset$  (as  $u_1$  is adjacent to  $s$  but not to  $t$ , none of its neighbors on  $C$  can be adjacent to  $s$ , and so on along the cycle) and that the neighbors of  $s$  and  $t$  are alternating on  $C'$ . Without loss of generality, we suppose that  $C' = u_1v_1u_2v_2u_3v_3$ . We consider now the five following sets of vertices of  $H$  :  $M_1 = \{u_1\}$ ,  $M_2 = \{s\}$ ,  $M_3 = \{u_2, v_1\}$ ,  $M_4 = \{v_2, u_3, v_3\}$ ,  $M_5 = \{t\}$ . They are depicted on [Figure 8](#). Let  $\mu: \hat{K}_4 \rightarrow \mathcal{P}^{<\omega}(V(H))$  be the function defined as follows:  $\forall i \in \llbracket 1, 5 \rrbracket$ ,  $\mu(v_i) = M_i$  (using the names of vertices of  $\hat{K}_4$  defined on [Figure 1](#)). Now, remark that  $\mu$  is a model of  $\hat{K}_4$  in  $H$ : for every  $i \in \llbracket 1, 5 \rrbracket$ , the set  $M_i$  is connected,  $M_1, M_3, M_4$  forms a cycle (using edges  $\{u_1, v_1\}, \{u_2, v_2\}, \{v_3, u_1\}$  in this order),  $M_2$  is

adjacent to any of these three sets (by edges  $\{s, u_1\}, \{s, u_2\}, \{s, u_3\}$ ) and the set  $M_5$  is only adjacent to  $M_3$  and  $M_4$  (by edges  $\{t, v_1\}, \{t, v_2\}$ ). Remark that the previous statement holds even when  $C'$  is not an induced cycle, as any possible chord of  $C'$  will be between vertices of the sets  $M_1, M_2, M_3, M_4$  (as  $M_5$  is reduced to  $t \notin V(C')$ ) and these sets are already all pairwise adjacent. We assumed our initial graph to be  $\hat{K}_4$ -induced minor-free but we proved that it contains a model of  $\hat{K}_4$ : this is the contradiction we were looking for.  $\square$

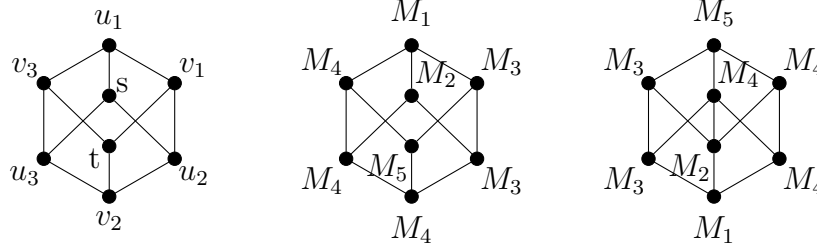


Figure 8: Graph  $H$  (left) used in Lemma 8 (middle) and in Lemma 9 (right).

**Lemma 9.** *Every two adjacent vertices of  $R$  have the same neighborhood in  $C$ .*

*Proof.* By contradiction, we assume that there are two adjacent vertices  $s, t \in V(R)$  and  $u_1 \in V(C)$  such that  $\{s, u_1\} \in E(G)$  but  $\{t, u_1\} \notin E(G)$ . As this proof is very similar to the proof of Lemma 8, we define  $u_1, u_2, u_3, v_1, v_2, v_3, U, V, H, C'$  in the same way here.

*First case:*  $C'$  is an induced cycle. In this case, the graph  $H$  is the (induced) cycle  $C' = u_1v_1u_2v_2u_3v_3$  plus the two adjacent vertices  $s$  and  $t$ . Let us define five vertex sets:  $M_1 = \{v_2\}, M_2 = \{t\}, M_3 = \{u_3, v_3\}, M_4 = \{v_1, u_2, s\}, M_5 = \{u_1\}$ . They are depicted on Figure 8. Now, remark that the function that sends the vertex of  $\hat{K}_4$  labeled  $i$  on Figure 1 to  $M_i$  is a model of  $\hat{K}_4$  in  $H$ : every set  $M_i$  is connected,  $M_1, M_3, M_4$  forms a cycle (using edges  $\{v_2, u_3\}, \{u_3, s\}, \{u_2, v_2\}$  is this order),  $M_2$  is adjacent to any of these three sets (by edges  $\{t, v_2\}, \{t, v_3\}, \{t, v_1\}$ ) and the set  $M_5$  is only adjacent to  $M_3$  and  $M_4$  (by edges  $\{u_1, v_3\}, \{u_1, v_1\}$ ). As we assumed our initial graph to be  $\hat{K}_4$ -induced minor-free, this is a contradiction.

*Second case:* the cycle  $C'$  is not induced. By Lemma 7 and as  $C$  is supposed to be a minimal 3-wheel of  $G$ , the cycle  $C$  has only one chord. In this case, the graph  $H$  is the (induced) cycle  $C' = u_1v_1u_2v_2u_3v_3$  plus the two adjacent vertices  $s$  and  $t$  and an edge  $e$  between two vertices of  $C'$ . In  $H$ , both  $H \setminus \{s\}$  and  $H \setminus \{t\}$  are minimal 3-wheels, sharing the same cycle  $C'$ . By applying Lemma 7 on these two 3-wheels, we obtain that the endpoints of  $e$  must both be adjacent to a vertex of  $u_1, u_2, u_3$  (neighbor of  $s$  on  $C'$ ) and to a vertex of  $v_1, v_2, v_3$  (neighbor of  $t$  on  $C'$ ). Such a configuration is impossible.  $\square$

**Lemma 10.** *If  $C$  is not an induced cycle of  $G$ , then  $|V(R)| = 1$ .*

*Proof.* Let  $r \in V(R)$  be the center of a minimum 3-wheel of cycle  $C$ . By contradiction, let us assume that  $R$  contains a vertex  $s \neq r$ . By Lemma 7,  $r$  has exactly three neighbors on  $C$ , one of which, that we call  $u$ , is adjacent to both endpoints of the only chord of  $C$ . Furthermore the two other neighbors of  $r$ , that we denote by  $\{v, w\}$ , are adjacent. According to Lemma 8 and Lemma 8,  $r$  and  $s$  are adjacent in  $G$ . There are now two different cases to consider depending whether  $\{r, s\}$  is an edge or not. The case  $\{r, s\} \in E(G)$  (respectively  $\{r, s\} \notin E(G)$ ) is depicted on the right (respectively left) of Figure 9. Remark that if  $\{r, s\} \in E(G)$ , the graph  $G$  satisfies conditions of Lemma 5 (with  $G[\{r, s, u, v, w\}]$  for  $S$ ,  $\{u, v\}$  for  $\{s_1, s_2\}$  and  $x$  for  $x$ ), so  $G \leq_{\text{im}} \hat{K}_4$ , what is contradictory. In the other hand, when  $\{r, s\} \notin E(G)$ , the induced subgraph  $G[\{r, s, u, v, w\}]$  is isomorphic to  $\hat{K}_4$ , so we also have  $G \leq_{\text{im}} \hat{K}_4$ . Hence  $|V(R)| < 2$ . By definition of  $C$  and  $R$ , the subgraph  $R$  contains at least one vertex (that is the center of a 3-wheel of cycle  $C$ ), thus  $|V(R)| = 1$  as required.

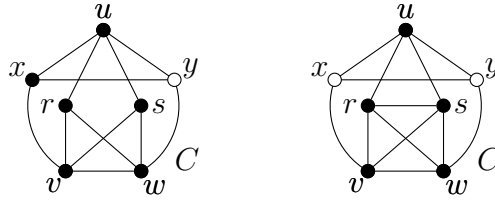


Figure 9: Two different cases in the proof of Lemma 10

□

**Corollary 4.** *If  $C$  is not an induced cycle of  $G$ , then  $G$  is a subdivision of the prism.*

**Lemma 11.**  *$R$  is complete multipartite.*

*Proof.* As a graph is complete multipartite iff it does not contain  $K_1 + K_2$  as induced subgraph, we only need to show that the case where  $R = K_1 + K_2$  is not possible. Consequently, let us assume that we are in this case, and let  $u, v, w$  be the vertices of  $R$ ,  $\{u, v\}$  being the only edge in  $R$ . As  $R$  is an induced subgraph of  $G$ ,  $u$  and  $w$  are not adjacent in  $G$  neither. By Lemma 8, they have the same neighborhood on  $C$ . The same argument can be applied to  $v$  and  $w$  to show that  $N_C(u) = N_C(v) = N_C(w)$ . Let  $x, y, z \in N_C(u)$  be three different vertices (they exist by Corollary 2) and let  $H$  be the graph obtained from  $G$  by contracting every edge of  $C$  that is not incident with two vertices of  $\{x, y, z\}$ . Such a graph is a triangle (obtained by contracting  $C$ ) and the three vertices  $u, v, w$  each adjacent to every vertex of the triangle, as drawn in Lemma 11. Deleting vertex  $y$  gives a graph isomorphic to  $\hat{K}_4$ , as one can easily check (cf. Lemma 11). □

We now need to show that every graph containing a  $K_4$ -subdivision either has a proper  $K_4$ -subdivision, or fall in the possible cases of the statement of Theorem 2.

**Lemma 12.** *If  $G$  can be obtained by adding an edge between two vertices of a  $K_{3,3}$ -subdivision, then  $G$  has a proper  $K_4$ -subdivision.*

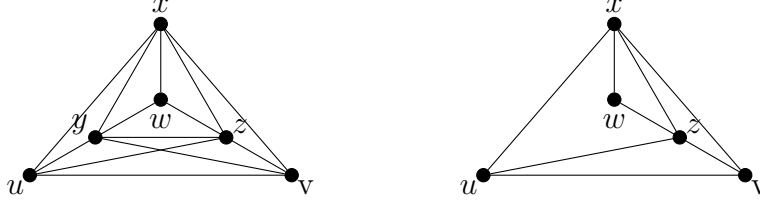


Figure 10: The graph  $H$  of [Lemma 11](#) (left) and the graph obtained after deletion of  $y$  (right).

*Proof.* Let  $S$  be the spanning subgraph of  $G$  which is a  $K_{3,3}$ -subdivision. A *branch* of  $S$  is a maximal path, the internal vertices of which are of degree two. In  $S$ , non-subdivision vertices are connected by branches. Let us call  $a, b, c, x, y, z$  the non-subdivision vertices of  $S$  in a way such that there is neither a branch between any two vertices of  $\{a, b, c\}$ , nor between any two vertices of  $\{x, y, z\}$  (intuitively  $\{a, b, c\}$  and  $\{x, y, z\}$  correspond to the two maximum independent sets of  $K_{3,3}$ ). Observe that every  $K_{3,3}$ -subdivision contains a  $K_4$ -subdivision (but not a proper one). Let us now consider all the possible endpoints of the only edge  $e$  of  $E(G) \setminus E(S)$ .

*First case:* both endpoints of  $e$  belong to the same branch  $B$  of  $S$ . Let  $X$  be the set of internal vertices of the subpath of  $B$  starting at the one endpoint of  $e$  and ending at the other one. As  $G$  is a simple graph,  $|X| \geq 1$ . Then  $G \setminus X$  has a  $K_4$ -subdivision (as it is a  $K_{3,3}$ -subdivision), which is a proper  $K_4$ -subdivision of  $G$ .

*Second case:*  $e$  is incident with two non-subdivision vertices. Observe that the case where  $e$  is incident with a vertex from  $\{a, b, c\}$  and the other from  $\{x, y, z\}$  is contained in the previous case. Let us assume without loss of generality that  $e = \{a, b\}$ . Then  $G \setminus \{x\}$  has a  $K_4$ -subdivision. Indeed, if  $B_{s,t}$  denotes the branch with endpoints the vertices  $s$  and  $t$  (for  $(s, t) \in \{a, b, c\} \times \{x, y, z\}$ ), then the vertices of the paths  $B_{b,y}$ ,  $B_{b,z}$ ,  $B_{c,z}$  and  $B_{c,y}$  induce a cycle in  $G$ . The vertex  $a$  is then connected to this cycle with the paths  $B_{a,y}$ ,  $B_{a,z}$  and the edge  $e$ . Hence  $G$  has a proper  $K_4$ -subdivision, as required.

*Third case:*  $e$  is incident with two subdivision vertices. If the two endpoints of  $e$  belong to the same branch, then we are in the first case. Otherwise, we can easily reach the second case as follows. If we contract all the edges on the path connecting the first endpoint of  $e$  to a vertex of  $\{a, b, c\}$  and all the edges on the path connecting the second endpoint of  $e$  to a vertex of  $\{x, y, z\}$ , we get a  $K_4$ -subdivision (because we never contracted an edge incident with two non-subdivision vertices of  $S$ ) plus the edge  $e$  which is now incident with two non-subdivision vertices. This concludes the proof.  $\square$

**Lemma 13.** *If  $G$  can be obtained by adding an edge between two vertices of a prism-subdivision, then  $G$  has a proper  $K_4$ -subdivision.*

*Proof.* Let  $S$  be a prism-subdivision in  $G$  and let  $e \in E(G) \setminus E(S)$ . We will use the concept of *branch* defined in the proof of [Lemma 12](#), which is very similar to this one. Let us call  $a, b, c, x, y, z$  the non-subdivision vertices in a way such that there are branches between every pair of vertices of  $\{a, b, c\}$  (respectively  $\{x, y, z\}$ ) and between

vertices of the pairs  $(a, x)$ ,  $(b, y)$ , and  $(c, z)$ . Intuitively  $\{a, b, c\}$  and  $\{x, y, z\}$  correspond to the two triangles of the prism. Let us consider the positions of the endpoints of  $e$ .

*First case:* both endpoints of  $e$  belong to the same branch of  $S$ . Since the prism contains a  $K_4$  subdivision (but not a proper one), we can in this case find a smaller prism subdivision as in the first case of the proof of [Lemma 12](#), and thus a proper  $K_4$ -subdivision.

*Second case:*  $e$  is incident with two non-subdivision vertices. Let us assume without loss of generality that  $e = \{a, y\}$  (the cases  $e \subseteq \{a, b, c\}$ ,  $e \subseteq \{x, y, z\}$ , and  $e \in \{\{a, x\}, \{b, y\}, \{c, z\}\}$  are subcases of the first one). Then in  $G \setminus \{x\}$ , the paths  $B_{a,b}$ ,  $B_{b,z}$  and  $B_{x,y}$  together with the edge  $e$  induces a cycle to which the vertex  $c$  is connected via the paths  $B_{c,b}$ ,  $B_{c,a}$ , and  $B_{c,y}$ . Hence  $G$  has a proper  $K_4$ -subdivision.

*Third case:*  $e$  is incident with two branches between  $a, b$ , and  $c$  (and the symmetric case with branches between  $x, y$ , and  $z$ ). Let us assume without loss of generality that  $e$  has one endpoint  $r$  among the interior vertices of  $B_{a,c}$  and the other one  $s$  among the interior vertices of  $B_{b,c}$ . Then  $(r, s)$ ,  $sB_{b,c}c$ , and  $cB_{a,c}r$  induces in  $G \setminus \{x\}$  a cycle to which the vertex  $b$  is connected via the paths  $bB_{b,c}s$ ,  $bB_{a,b}a$  together with  $aB_{a,c}r$ , and  $B_{b,z}$  together with  $B_{z,y}$  and  $B_{y,b}$ . Again,  $G$  has a proper  $K_4$ -subdivision.

*Fourth case:*  $e$  is incident with two branches, the one connected to a vertex in  $\{a, b, c\}$  and the other one connected to a vertex in  $\{x, y, z\}$ . In this case, by contracting the edges of the first branch that are between the endpoint of  $e$  and a vertex of  $\{a, b, c\}$  and similarly with the second branch and a vertex of  $\{x, y, z\}$  gives a graph with a prism-subdivision plus an edge between two non-subdivision vertices (that is, first case), exactly as in the proof of [Lemma 12](#).  $\square$

**Lemma 14.** *If graph  $G$  contains a  $K_4$ -subdivision, then either  $G$  has a proper  $K_4$ -subdivision, or  $G$  is a wheel, or a subdivision of one of the following graphs:  $K_4$ ,  $K_{3,3}$ , and the prism.*

*Proof.* Looking for a contradiction, let  $G$  be a counterexample with the minimum number of vertices and, subject to that, the minimum number of edges. Let  $S$  be a  $K_4$ -subdivision in  $G$  and let  $e \in E(G) \setminus E(S)$ . Observe that since  $G$  has no proper  $K_4$ -subdivision,  $S$  is a spanning subgraph of  $G$ . Also,  $e$  is well defined as we assume that  $G$  is not a  $K_4$ -subdivision. Notice that since the minimum degree of  $K_4$  is 3, contracting an edge incident with a vertex of degree 2 in  $G$  would yield a smaller counterexample. Therefore  $G$  has minimum degree at least 3. Let  $G' = G \setminus \{e\}$ . This graph clearly contains  $S$ . By minimality of  $G$ , the graph  $G'$  is either a wheel, or a subdivision of a graph among  $K_4$ ,  $K_{3,3}$ , and the prism. Observe that  $G'$  cannot have a proper  $K_4$ -subdivision because it would also be a proper  $K_4$ -subdivision in  $G$ .

*First case:*  $G'$  is a wheel. Let  $C$  be the cycle of the wheel and let  $r$  be its center. Obviously, in  $G$  the edge  $e$  has not  $r$  as endpoint otherwise  $G$  would also be a wheel. Therefore  $e$  is incident with two vertices of  $C$ . Let  $P$  and  $P'$  be the two subpaths of  $C$  whose endpoints are then endpoints of  $e$ . Observe that none of  $P$  and  $P'$  contains more than two neighbors of  $r$ . Indeed, if, say,  $P$  contained at least three neighbors of  $r$ , then the subgraph of  $G$  induced by the vertices of  $P$ ,  $e$ , and  $r$  would contain a  $K_4$ -subdivision, hence contradicting the fact that  $G$  has no proper  $K_4$ -subdivision.

Therefore  $G$  is the cycle  $C$  with exactly one chord,  $e$ , and the vertex  $r$  which has at most 4 neighbors on  $C$ . Because  $G$  has maximum degree at least 3, it has at most 7 vertices. If  $r$  has three neighbors on  $C$ , then necessarily  $P$  contains one of them and  $P'$  the other two (or the other way around). We can easily check in this case that  $G$  is a subdivision of the prism. If  $r$  has four neighbors on  $C$ , the interior of  $P$  and  $P'$  must each contain two of them according to the above remarks. The deletion of any neighbor of  $r$  in this graph yields a  $K_4$ -subdivision of non-subdivision vertices  $r$  and the remaining neighbors. Observe that both cases contradict the assumptions made on  $G$ .

*Second case:*  $G'$  is a subdivision of  $K_4$ , or  $K_{3,3}$ , or the prism. If  $G'$  is a subdivision of  $K_{3,3}$  or of the prism, then the result follows by [Lemma 12](#) and [Lemma 13](#). Let us now assume that  $G'$  is a subdivision of  $K_4$  and let us consider branches of this subdivision as defined in the proof of [Lemma 12](#). If  $e$  has endpoints in the same branch, then as in the first cases of the aforementioned lemmas we can find in  $G'$  a  $K_4$ -subdivision with fewer vertices and thus a proper  $K_4$ -subdivision in  $G$ . In the case where the endpoints of  $e$  belong to the interior of two different branches, then it is easy to see that  $G$  is a prism-subdivision. Let  $\{x, y, z, t\}$  be the non-subdivision vertices of the  $K_4$ -subdivision. Finally, let us assume that the one endpoint of  $e$  is a non-subdivision vertex, say  $x$ , and the other one, that we call  $u$ , is a subdivision vertex of a branch, say  $B_{y,z}$  (using the same notation as in the proof of [Lemma 12](#)). If  $X$  is set of interior vertices of one of  $B_{x,y}$ ,  $B_{x,z}$ , or  $B_{x,t}$ , then the graph  $G \setminus X$  has a  $K_4$  subdivision of non-subdivision vertices  $x, u, z, t$ ,  $x, y, y, t$  or  $x, y, u, z$  respectively. In this case  $G$  has a proper  $K_4$ -subdivision. If none of  $B_{x,y}$ ,  $B_{x,z}$ , and  $B_{x,t}$  has internal vertices, then  $G$  is a wheel of center  $x$ .

In all the possible cases we reached the contradiction we were looking for. This concludes the proof.  $\square$

We are now ready to prove [Theorem 2](#).

*Proof of Theorem 2.* Let  $G \in \text{Excl}_{\text{im}}(\hat{K}_4)$  be a biconnected graph. If  $G$  does not contain a  $K_4$  subdivision, then the theorem is trivially true for  $G$ . If graph  $G$  contains a  $K_4$ -subdivision but not a proper one, from [Lemma 14](#) we get that  $G$  is a subdivision of one of  $K_4$ ,  $K_{3,3}$ , or the prism, in which case the theorem holds, or that  $G$  is a wheel, which has a trivial cycle-multipartite decomposition, with the center being the multipartite part.

Finally, let us assume that  $G$  contains a proper  $K_4$ -subdivision. By [Lemma 6](#),  $G$  contains a 3-wheel. Let  $C$  be the cycle of a minimum 3-wheel in  $G$  and  $R$  the subgraph of  $G$  induced by  $V(G) \setminus V(C)$ . According to [Corollary 4](#), if  $C$  is not an induced cycle, then  $G$  is a subdivision of the prism. When  $C$  is induced, then by [Lemma 11](#),  $R$  is complete. Furthermore, vertices of  $R$  have the same neighborhood on  $C$ , as proved in [Lemma 8](#) and [Lemma 9](#). Therefore,  $(C, R)$  is a cycle-multipartite decomposition of  $G$  and we are done.  $\square$

## 6.2 From a decomposition theorem to well-quasi-ordering

This section is devoted to the proof of [Theorem 4](#).



The proof relies on the two following lemmas which are proved in the next subsections.

**Lemma 15.** *For every (unlabeled) graph  $G$  and every wqo  $(S, \preceq)$ , the class of  $(S, \preceq)$ -labeled  $G$ -subdivisions is well-quasi-ordered by the contraction relation.*

**Lemma 16.** *For every wqo  $(S, \preceq)$ , the class of  $(S, \preceq)$ -labeled graphs having a cycle-multipartite decomposition is well-quasi-ordered by induced minors.*

We also use the following result by Thomas.

**Proposition 4** ([Tho85]). *The class of  $K_4$ -induced minor-free graphs is wqo by  $\leq_{\text{im}}$ .*

*Proof of Theorem 4.* The class of graphs not containing  $K_4$  as minor (or, equivalently, as induced minor) has been shown to be well-quasi-ordered by induced minors in [Tho85], cf. Proposition 4. According to Remark 1, we can then restrict our attention to graphs of  $\text{Excl}_{\text{im}}(\hat{K}_4)$  that contain  $K_4$  as minor. As some of these graphs might not be biconnected, we use Proposition 1: it is enough to show that for every wqo  $(S, \preceq)$ , the class of  $(S, \preceq)$ -labeled biconnected graphs containing  $K_4$  as minor are wqo by induced minors. By Theorem 2, this class can be divided into two subclasses:

- (biconnected) subdivisions of a graph among  $K_4$ ,  $K_{3,3}$ , and the prism;
- graphs having a cycle-multipartite decomposition.

Lemma 15 and Lemma 16 handle these two cases, hence by Remark 1 the class of  $(S, \preceq)$ -labeled biconnected graphs containing  $K_4$  as minor are wqo by induced minors for every wqo  $(S, \preceq)$ . This concludes the proof.  $\square$

The following subsections contains the proofs of Lemma 15 and Lemma 16. The technique that we repeatedly use in order to show that a poset  $(A, \preceq_A)$  is a wqo is the following:

1. we define a function  $f: A' \rightarrow A$ . Intuitively, elements of  $A'$  can be seen as descriptions (or encodings) of objects of  $A$  and  $f$  is the function constructing the objects from the descriptions;
2. we show that  $A'$  is wqo by some relation  $\preceq_{A'}$ . Usually,  $A'$  is a product, union or sequence over known wqos so this can be done using Proposition 2 and Proposition 3 together with Remark 1;
3. we prove that  $f$  an epi of  $(A', \preceq_{A'}) \rightarrow (A, \preceq_A)$  (sometimes using Remark 3) and by Remark 2 we conclude that  $(A, \preceq_A)$  is a wqo.

## 6.3 Well-quasi-ordering subdivisions

Let  $\mathcal{OP}$  denote the class of paths whose endpoints are distinguished, i.e. one end is said to be the beginning and the other one the end. In the sequel,  $\text{fst}(P)$  denotes the first vertex of the path  $P$  and  $\text{lst}(P)$  its last vertex. We extend the relation  $\leq_{\text{im}}$  to  $\mathcal{OP}$  as follows: for every  $G, H \in \mathcal{OP}$ ,  $G \leq_{\text{im}} H$  if there in an induced minor model  $\mu$  of  $G$  in  $H$  such that  $\text{fst}(H) \in \mu(\text{fst}(G))$  and  $\text{lst}(H) \in \mu(\text{lst}(G))$ , and similarly for  $\leq_c$ .

**Lemma 17.** *If the poset  $(Q, \preceq_Q)$  is a wqo, then the poset  $(\text{lab}_{(Q, \preceq_Q)}(\mathcal{OP}), \leq_c)$  also is a wqo.*

*Proof.* Let us take  $(S, \preceq)$  to be  $(\mathcal{P}^{<\omega}(Q), \preceq_Q^{\mathcal{P}})$ . By [Corollary 1](#)  $(S, \preceq)$  is a wqo. We consider the function  $f: (S^*, \preceq^*) \rightarrow (\text{lab}_{(S, \preceq)}(\mathcal{OP}), \leq_c)$  that, given a sequence  $\langle s_1, \dots, s_k \rangle \in S^*$  of elements of  $S$ , returns the path  $P$  on  $k$  vertices whose  $i$ -th vertex  $v_i$  is labeled by  $s_i$  for every  $i \in \llbracket 1, k \rrbracket$  and where  $\text{fst}(P) = v_1$  and  $\text{lst}(P) = v_k$ . The image of this function is clearly  $\text{lab}_{(S, \preceq)}(\mathcal{OP})$  and by [Proposition 3](#) its domain is well-quasi-ordered by  $\preceq^*$ . By the virtue of [Remark 2](#), it is thus enough to show that  $f: (S^*, \preceq^*) \rightarrow (\text{lab}_{(S, \preceq)}(\mathcal{OP}), \leq_c)$  is monotone in order to prove that  $(\text{lab}_{(S, \preceq)}(\mathcal{OP}), \leq_c)$  is wqo.

Let  $R = \langle r_1, \dots, r_k \rangle, S = \langle s_1, \dots, s_l \rangle \in S^*$ , be two sequences such that  $R \preceq^* S$  and let us show that  $f(R) \leq_c f(S)$ . We will use the following notation:  $f(R)$  is the path  $v_1 \dots v_k$  labeled by  $\lambda_R$  and similarly for  $f(S)$ ,  $u_1 \dots u_l$  and  $\lambda_S$ . Let  $\varphi: \llbracket 1, k \rrbracket \rightarrow \llbracket 1, l \rrbracket$  be an increasing function such that  $\forall i \in \llbracket 1, k \rrbracket$ , we have  $r_i \preceq s_{\varphi(i)}$  (such a function exists since  $R \preceq^* S$ ). Let us consider the path obtained from  $f(S)$  by, for every  $i \notin \{\varphi(j)\}_{j \in \llbracket 1, k \rrbracket}$ , contracting the label of  $u_i$  to the empty set and then dissolving  $u_i$ . Remark that this graph is a path on  $k$  vertices  $p_1 p_2 \dots p_k$  such that  $\forall i \in \llbracket 1, k \rrbracket$ ,  $\lambda_R(v_i) = r_i \preceq \lambda_S(p_i) = s_{\varphi(i)}$ . Furthermore, this path is a contraction of  $f(S)$  where either  $u_1 = p_1$  (respectively  $u_l = p_k$ ) or this vertex has been contracted to  $p_1$  (respectively  $p_k$ ), hence  $f(R) \leq_c f(S)$  as desired.  $\square$

*Proof of Lemma 15.* Let  $G$  be a non labeled graph, let  $(S, \preceq)$  be a wqo and let  $\mathcal{G}$  be the class of all  $(S, \preceq)$ -labeled  $G$ -subdivisions. We set  $m = |E(G)|$ . Let us show that  $(\mathcal{G}, \leq_c)$  is a wqo. First, we arbitrarily choose an orientation to every edge of  $G$  and an enumeration  $e_1, \dots, e_m$  of these edges. We now consider the function  $f$  that, given a tuple  $(Q_1, \dots, Q_m)$  of  $m$  paths of  $\text{lab}_{(S, \preceq)}(\mathcal{OP})$ , returns the graph constructed from  $G$  by, for every  $i \in \llbracket 1, m \rrbracket$ , replacing the edge  $e_i$  by the path  $Q_i$ , while respecting the orientation, i.e. the first (respectively last) vertex of  $Q_i$  goes to the first (respectively last) vertex of  $e_i$ . By [Proposition 2](#) on Cartesian product of wqos and since  $(\text{lab}_{(S, \preceq)}(\mathcal{OP}), \leq_c)$  is a wqo ([Lemma 17](#)), the domain  $\text{lab}_{(S, \preceq)}(\mathcal{OP})^m$  of  $f$  is well-quasi-ordered by  $\leq_c^m$ . Notice that every element of the codomain of  $f$  is an  $G$ -subdivision (by definitions of  $f$ ), and moreover that  $f$  is surjective on  $\mathcal{G}$ : for every  $(S, \preceq)$ -labeled  $G$ -subdivision  $H$  we can construct a tuple  $(Q_1, \dots, Q_m)$  of  $m$  paths of  $\text{lab}_{(S, \preceq)}(\mathcal{OP})$ , such that  $f(Q_1, \dots, Q_m) = H$ .

In order to show that  $(\mathcal{G}, \leq_c)$  is a wqo, it is enough to prove that  $f: (\text{lab}_{(S, \preceq)}(\mathcal{OP}), \leq_c^m) \rightarrow (\mathcal{G}, \leq_c)$  is an epi, as explained in [Remark 2](#), that is, to prove that for every two tuples  $\mathcal{Q}, \mathcal{R} \in \text{lab}_{(S, \preceq)}(\mathcal{OP})^m$  such that  $\mathcal{Q} \leq_c^m \mathcal{R}$ , we have  $f(\mathcal{Q}) \leq_c f(\mathcal{R})$ . According to [Remark 3](#), we only need to care, for every  $i \in \llbracket 1, m \rrbracket$ , of the case where  $\mathcal{Q}$  and  $\mathcal{R}$  only differs by the  $i$ -th coordinate. It is at this point of the proof important to remark the symmetry of the definition of  $f$ : since the different coordinates any element of the domain of  $f$  are playing the same role, we only have to deal with the case where  $\mathcal{Q}$  and  $\mathcal{R}$  differs by one (fixed) coordinate, say the first one. Therefore, let us consider two tuples  $\mathcal{Q} = (Q, Q_2, \dots, Q_m)$  and  $\mathcal{R} = (R, Q_2, \dots, Q_m)$  of  $\text{lab}_{(S, \preceq)}(\mathcal{OP})^m$  such that  $\mathcal{Q} \leq_c^m \mathcal{R}$ , i.e. satisfying  $Q \leq_c R$ . Let  $\mu: V(Q) \rightarrow \mathcal{P}^{<\omega}(V(R))$  be a contraction model of  $Q$  in  $R$  and let  $\mu': V(f(\mathcal{Q})) \rightarrow \mathcal{P}^{<\omega}(V(f(\mathcal{R})))$  be the trivial contraction model of



$f(\mathcal{Q}) \setminus V(Q)$  in itself defined by  $\forall u \in V(f(\mathcal{Q})) \setminus V(Q)$ ,  $\mu'(u) = \{u\}$ . We now consider the function  $\nu: V(f(\mathcal{Q})) \rightarrow \mathcal{P}^{<\omega}(V(f(\mathcal{R})))$  defined as follows:

$$\nu: \begin{cases} V(f(\mathcal{Q})) & \rightarrow \mathcal{P}^{<\omega} V(f(\mathcal{R})) \\ u & \mapsto \mu(u) \quad \text{if } u \in V(Q) \\ u & \mapsto \mu'(u) \quad \text{otherwise.} \end{cases}$$

Let us show that  $\nu$  is a contraction model of  $f(\mathcal{Q})$  in  $f(\mathcal{R})$ . First, notice that since both  $\mu$  and  $\mu'$  are contraction models,  $\nu$  inherits some of their properties: for every  $u \in V(f(\mathcal{Q}))$ , the induced subgraph  $V(f(\mathcal{R}))[u]$ , is connected and  $\lambda_{f(\mathcal{Q})}(u) \subseteq \bigcup_{u \in \mu(u)} \lambda_{f(\mathcal{R})}(u)$ . For the same reason, we have:

$$\begin{aligned} \bigcup_{u \in V(f(\mathcal{Q}))} \nu(u) &= \bigcup_{u \in V(Q)} \mu(u) \cup \bigcup_{u \in V(f(\mathcal{Q}) \setminus V(Q))} \mu'(u) \\ &= V(R) \cup V(f(\mathcal{Q})) \setminus V(Q) \\ &= V(f(\mathcal{R})). \end{aligned}$$

Let us now consider two distinct vertices  $u$  and  $v$  of  $f(\mathcal{Q})$ .

*First case:*  $u$  and  $v$  both belong to the same set among  $V(Q)$  and  $V(f(\mathcal{Q})) \setminus V(Q)$ . In this case  $\nu(u)$  and  $\nu(v)$  are disjoint, and they are adjacent iff  $\{u, v\} \in E(f(\mathcal{Q}))$  since both  $\mu$  and  $\mu'$  are contraction models.

*Second case:*  $u \in V(Q)$  and  $v \in V(f(\mathcal{Q})) \setminus V(Q)$  (or the symmetric case). As in the previous case,  $\nu(u)$  and  $\nu(v)$  are disjoint. Assume that  $\{u, v\}$  is an edge of  $f(\mathcal{Q})$ . Notice that we necessarily have either  $u = \text{fst}(Q)$  and  $v \in N_{f(\mathcal{Q}) \setminus V(Q)}(\text{fst}(Q))$ , or  $u = \text{lst}(Q)$  and  $v \in N_{f(\mathcal{Q}) \setminus V(Q)}(\text{lst}(Q))$ . Let us assume without loss of generality that we are in the first of these two subcases. By definition of  $f(\mathcal{R})$ ,  $\{\text{fst}(R), v\}$  is an edge. Since  $\mu$  is a contraction model, we then also have  $\text{fst}(R) \in \nu(u)$  and therefore  $\nu(v)$  and  $\nu(u)$  are adjacent in  $f(\mathcal{R})$ .

We just proved that  $\nu$  is a model of  $f(\mathcal{Q})$  in  $f(\mathcal{R})$ . As explained above, this is enough in order to show that  $f$  is monotone with regard to  $\leq_c^m, \leq_c$ . Hence  $(\mathcal{G}, \leq_c)$  is a wqo and this concludes the proof.  $\square$

## 6.4 Well-quasi-ordering cycle-multipartite decompositions

In this section, we show that graphs having a cycle-multipartite decomposition are well-quasi-ordered by induced minors.

**Lemma 18.** *If  $(Q, \preceq_Q)$  is wqo then the class of  $(Q, \preceq_Q)$ -labeled independent sets is wqo by the induced subgraph relation.*

*Proof.* We will again define  $(S, \preceq) := (\mathcal{P}^{<\omega}(Q), \preceq_Q^{\mathcal{P}})$ , and observe that it is a wqo.

The function  $f$  that maps every sequence  $\langle x_1, \dots, x_k \rangle$  (for some positive integer  $k$ ) of elements of  $S$  to the  $(S, \preceq)$ -labeled independent set on vertex set  $\{v_1, \dots, v_k\}$  where  $v_i$  have label  $x_i$  for every  $i \in \llbracket 1, k \rrbracket$  has clearly the class of  $(S, \preceq)$ -labeled independent sets as codomain. Let us show that  $f$  is an epi. Let  $X = \langle x_1, \dots, x_k \rangle, Y = \langle x_1, \dots, x_l \rangle \in S^*$  be two sequences such that  $X \preceq^* Y$ . By definition of the relation  $\preceq^*$ , there is an increasing function  $\varphi: \llbracket 1, k \rrbracket \rightarrow \llbracket 1, l \rrbracket$  such that  $\forall i \in \llbracket 1, k \rrbracket, x_i \preceq_{\varphi i}$ . Therefore the

function  $\mu: V(f(X)) \rightarrow V(f(Y))$  that maps the vertex  $v_i$  of  $f(X)$  to the singleton  $\{v_{\varphi(i)}\}$  of  $f(Y)$  is an induced subgraph model of  $f(X)$  in  $f(Y)$  and this proves the monotonicity of  $f$  with regard to  $\preceq^*, \leq_{\text{im}}$ . By the virtue of [Remark 2](#) and since  $(S^*, \preceq^*)$  is a wqo, we get that the class of  $(S, \preceq)$ -labeled independent sets is wqo by the induced subgraph relation.  $\square$

**Corollary 5.** *With a very similar proof, we can also show that if  $(S, \preceq)$  is wqo then the class of  $(S, \preceq)$ -labeled cliques is wqo by the induced subgraph relation.*

**Corollary 6.** *If a class of  $(S, \preceq)$ -labeled graphs  $(\mathcal{G}, \leq_{\text{im}})$  is wqo, then so is its closure by finite disjoint union (respectively join).*

*Proof.* Let  $\mathcal{U}$  be the closure of  $(\mathcal{G}, \leq_{\text{im}})$  by disjoint union. Remark that every graph of  $\mathcal{U}$  can be partitioned in a family of pairwise non-adjacent graphs of  $\mathcal{G}$ . Therefore we can define a function mapping every  $\mathcal{G}$ -labeled independent set to the graph of  $\mathcal{U}$  obtained from  $G$  by replacing each vertex by its label (which is an  $(S, \preceq)$ -labeled graph). It is easy to check that this function is an epi of  $(\mathcal{G}, \leq_{\text{im}}) \rightarrow (\mathcal{U}, \leq_{\text{im}})$ . Together with [Remark 2](#) and [Lemma 18](#), this yields the desired result.  $\square$

**Corollary 7.** *If  $(S, \preceq)$  is a wqo then the class of  $(S, \preceq)$ -labeled complete multipartite graphs are wqo by the induced subgraph relation.*

*Proof of Lemma 16.* We consider the function  $f: (\text{lab}_{(S, \preceq)}(\mathcal{OP})^* \times \text{lab}_{(S, \preceq)}(\mathcal{K}_{\mathbb{N}^*}), \leq_c^* \times \leq_{\text{isg}}) \rightarrow (\text{lab}_{(S, \preceq)}(\mathcal{W}), \leq_{\text{im}})$  that, given a sequence  $[R_0, \dots, R_{k-1}] \in \text{lab}_{(S, \preceq)}(\mathcal{OP})$  of  $(S, \preceq)$ -labeled paths of  $\mathcal{OP}$  and a  $(S, \preceq)$ -labeled complete multipartite graph  $K$ , returns the graph constructed as follows.

1. consider the disjoint union of  $K$  and the paths of  $\{R_i\}_{i \in \llbracket 0, k-1 \rrbracket}$  and call  $v_i$  the vertex obtained by identifying the two vertices  $\text{lst}(R_i)$  and  $\text{fst}(R_{(i+1) \bmod k})$ , for every  $i \in \llbracket 0, k-1 \rrbracket$  (informally, this graph is the disjoint union of  $K$  and the cycle built by putting  $R_i$ 's end-to-end);
2. for every element  $v$  of  $\{v_i\}_{i \in \llbracket 0, k-1 \rrbracket}$ , add all possible edges between  $v$  and the vertices of  $K$ .

Remark that the codomain of  $f$  is  $\mathcal{W}$ . Indeed, every element of the image of  $f$  has a cycle-multipartite decomposition (by construction) and conversely, if  $G \in \mathcal{W}$  is of cycle-multipartite decomposition  $(C, K)$ , one can construct a sequence of  $R_0, \dots, R_k$  of subpaths of  $C$  meeting only on endpoints and whose interior vertices are of degree two such that  $G = f(R, \dots, R_{k-1}, K)$ . Let us show that the domain of  $f$  is well-quasi-ordered by  $\leq_c^* \times \leq_{\text{isg}}$ . We proved in [Lemma 17](#) that  $(\text{lab}_{(S, \preceq)}(\mathcal{OP}), \leq_c)$  is a wqo and [Corollary 7](#) shows that  $(\mathcal{K}_{\mathbb{N}^*}, \leq_{\text{isg}})$  is a wqo, so by applying [Proposition 3](#) we get first that  $(\text{lab}_{(S, \preceq)}(\mathcal{OP})^*, \leq_c^*)$  is a wqo, and then by [Proposition 2](#) together with [Corollary 7](#) that  $(\text{lab}_{(S, \preceq)}(\mathcal{OP})^* \times \mathcal{K}_{\mathbb{N}^*}, \leq_c^* \times \leq_{\text{isg}})$  is a wqo.

According to [Remark 2](#), it is enough to show that  $f: (\text{lab}_{(S, \preceq)}(\mathcal{OP})^* \times \text{lab}_{(S, \preceq)}(\mathcal{K}_{\mathbb{N}^*}), \leq_c^* \times \leq_{\text{isg}}) \rightarrow (\mathcal{W}, \leq_{\text{im}})$  is an epi in order to prove that  $(\text{lab}_{(S, \preceq)}(\mathcal{W}), \leq_{\text{isg}})$  is wqo. We will show the monotonicity of  $f$  in two steps: the first by showing that  $\forall R \in \text{lab}_{(S, \preceq)}(\mathcal{OP})^*, \forall H, H' \in \text{lab}_{(S, \preceq)}(\mathcal{K}_{\mathbb{N}^*}), H \leq_{\text{isg}} H' \Rightarrow f(R, H) \leq_{\text{im}} f(R, H')$  and the second by proving that  $\forall Q, R \in \text{lab}_{(S, \preceq)}(\mathcal{OP})^*, \forall H \in \text{lab}_{(S, \preceq)}(\mathcal{K}_{\mathbb{N}^*}), Q \leq_c^* R \Rightarrow$

$f(Q, H) \leq_{\text{im}} f(R, H)$ . According to [Remark 3](#), the desired result follows from these two assertions.

*First step.* Let  $R = \langle R_0, \dots, R_{k-1} \rangle \in \text{lab}_{(S, \preceq)}(\mathcal{OP})^*$  and  $H, H' \in \text{lab}_{(S, \preceq)}(\mathcal{K}_{\mathbb{N}^*})$  be such that  $H \leq_{\text{isg}} H'$ . We therefore have  $(R, H) \leq_c^* \times \leq_{\text{isg}} (R, H')$ . Let us show that  $f(R, H) \leq_{\text{im}} f(Q, H')$ . Since  $H \leq_{\text{isg}} H'$ , there is a subset  $A \subseteq V(H')$  such that  $H' \setminus A = H$ . Let  $C$  denote the cycle obtained from the disjoint union of  $\{R_i\}_{i \in \llbracket 0, k-1 \rrbracket}$  by the identification of the two vertices  $\text{lst}(R_i)$  and  $\text{fst}(R_{(i+1) \bmod k})$ , for every  $i \in \llbracket 0, k-1 \rrbracket$ , where we call  $v_i$  the vertex resulting from this identification. Let us consider the graph  $f(Q, H) \setminus A$ : to construct this graph we started with the disjoint union of  $C$  and  $H$ , then added all possible edges between  $v_i$  and  $V(H)$  for every  $i \in \llbracket 0, k-1 \rrbracket$  and at last deleted the vertices of  $A \setminus H$ . Remark that this graph is isomorphic to  $f(R, H)$ , and therefore  $f(R, H) \leq_{\text{im}} f(R, H')$ , as desired.

*Second step.* Let  $Q = \langle Q_0, \dots, Q_{k-1} \rangle$  and  $R = \langle R_0, \dots, R_{l-1} \rangle$  be two elements of  $\text{lab}_{(S, \preceq)}(\mathcal{OP})^*$  such that  $Q \leq_c^* R$  and let  $H \in \text{lab}_{(S, \preceq)}(\mathcal{K}_{\mathbb{N}^*})$ . We thus have  $(Q, H) \leq_c^* \times \leq_{\text{isg}} (R, H)$ . Let us show that  $f(Q, H) \leq_{\text{im}} f(R, H)$ . By definition of the relation  $\leq_c^*$ , there is an increasing function  $\varphi: \llbracket 0, k-1 \rrbracket \rightarrow \llbracket 0, l-1 \rrbracket$  such that

$$\forall i \in \llbracket 0, k-1 \rrbracket, Q_i \leq_c R_{\varphi(i)}.$$

For every  $i \in \llbracket 0, k-1 \rrbracket$ , let  $\mu_i: V(Q_i) \rightarrow \mathcal{P}^{<\omega}(R_{\varphi(i)})$  be a contraction model of  $Q_i$  in  $R_i$ . Recall that since  $Q_i$  and  $R_{\varphi(i)}$  are oriented paths, the contraction sending  $R_{\varphi(i)}$  on  $Q_i$  preserves endpoints. We now consider the function  $\mu$  defined as follows

$$\mu \begin{cases} V(f(Q, H)) & \rightarrow \mathcal{P}^{<\omega}(V(f(R, H))) \\ x & \rightarrow \{x\} \quad \text{if } x \in V(H) \\ \text{lst}(Q_i) & \rightarrow \mu_i(\text{lst}(Q_i)) \cup \bigcup_{j=\varphi(i)+1}^{\varphi(i+1)-1} V(R_j) \setminus \{\text{fst}(R_{\varphi(i+1)})\} \\ x & \rightarrow \mu_i(x) \subseteq R_{\varphi(i)} \quad \text{if } x \in Q_i \setminus \{\text{lst}(Q_i)\}. \end{cases}$$

We will show that  $\mu$  is an induced minor model of  $f(Q, H)$  in  $f(R, H)$ . First at all, remark that every element of the image of  $f$  induces in  $f(R, H)$  a connected subgraph:

- either  $x \in V(H)$  and  $\mu(x)$  is a singleton;
- or  $x \in \text{lst}(Q_i) \setminus \{\text{lst}(Q_i)\}$  and  $f(R, H)[\mu(x)]$  is connected since  $\mu_i(x) = \mu(x)$  is an induced minor model;
- or  $x = \text{lst}(Q_i)$  and  $\mu_i(\text{lst}(Q_i)) \cup \bigcup_{j=\varphi(i)+1}^{\varphi(i+1)-1} V(R_j) \setminus \{\text{fst}(R_{\varphi(i+1)})\}$  induces a in  $f(R, H)$  a connected subgraph because  $f(R, H)[\mu_i(\text{lst}(Q_i))]$  is connected and the other vertices are consecutive on the cycle.

Let us now show that adjacencies are preserved by  $\mu$ . Let  $u, v$  be two distinct vertices of  $f(Q, H)$ . If  $u, v \in H$ , then  $\mu(u)$  and  $\mu(v)$  are adjacent in  $f(R, H)$  iff  $u$  and  $v$  are in  $f(Q, H)$ , as  $\mu(u) = \{u\}$  and  $\mu(v) = \{v\}$  (informally, the “ $H$ -part” of  $f(R, H)$  is not changed by the model). If  $u, v \in Q$ , observe that  $u$  and  $v$  are adjacent in  $f(Q, H)$  iff they belong to the same path of  $\{Q_i\}_{i \in \llbracket 0, k-1 \rrbracket}$ . Thus in this case, the property that  $u$  and  $v$  are adjacent in  $f(Q, H)$  iff  $\mu(u)$  is adjacent to  $\mu(v)$  in  $f(R, H)$  is given by the fact that  $\{\mu_i\}_{i \in \llbracket 0, k-1 \rrbracket}$  are contraction models.

If  $u \in Q$  and  $v \in H$ , then  $\{u, v\} \in f(Q, H)$  (resp.  $\mu(u)$  is adjacent to  $\mu(v)$  in  $f(R, H)$ ) iff  $u$  is an endpoint of a path of  $\{Q_i\}_{i \in \llbracket 0, k-1 \rrbracket}$  (resp.  $\mu(u)$  contains an endpoint

of a path of  $\{R_i\}_{i \in \llbracket 0, l-1 \rrbracket}$ , by definition of  $f$ . As the contraction relation on oriented paths of  $\mathcal{OP}$  is required to contract endpoints to endpoints, the image  $\mu(u)$  must contain the endpoint of a path of  $\{R_i\}_{i \in \llbracket 0, l-1 \rrbracket}$  iff  $u$  is the endpoint of a path of  $\{Q_i\}_{i \in \llbracket 0, k-1 \rrbracket}$ . Therefore  $u$  and  $v$  are adjacent in  $f(Q, H)$  iff  $\mu(u)$  is adjacent to  $\mu(v)$  in  $f(R, H)$ , as required. We finally proved that  $f$  is monotone with regard to  $\leq_c^* \times \leq_{\text{isg}}, \leq_{\text{im}}$ . This was the only missing step in order to prove that  $(\text{lab}_{(S, \preceq)}(\mathcal{W}), \leq_{\text{isg}})$  is a wqo.  $\square$

## 7 Graphs not containing Gem

The purpose of this section to give a proof to [Theorem 5](#). This will be done by first proving a decomposition theorem for graphs of  $\text{Excl}_{\text{im}}(\text{Gem})$ , and then using this theorem to prove that  $(\text{Excl}_{\text{im}}(\text{Gem}), \leq_{\text{im}})$  is a wqo.

### 7.1 A Decomposition theorem for $\text{Excl}_{\text{im}}(\text{Gem})$

This section is devoted to the proof of [Theorem 3](#), which is split in several lemmas. In the sequel,  $G$  is a 2-connected graph of  $\text{Excl}_{\text{im}}(\text{Gem})$ . When  $G$  is 3-connected, we will rely on the following result originally proved by Ponomarenko.

**Proposition 5** ([\[Pon91\]](#)). *Every 3-connected Gem-induced minor-free graph is either a cograph, or has an induced subgraph  $S$  isomorphic to  $P_4$ , such that every connected component of  $G \setminus S$  is a cograph.*

Therefore we will here focus on the case where  $G$  is 2-connected but not 3-connected. A *rooted diamond* is a graph which can be constructed from a rooted  $C_4$  by adding a chord incident with exactly one endpoint of the root (cf. [Figure 11](#)).



Figure 11: A rooted diamond, the root being the thick edge.

**Lemma 19.** *Let  $S = \{v_1, v_2\}$  be a cutset in  $G$  and let  $C$  be a component of  $G \setminus S$ . Let  $H$  be the graph  $G[V(C) \cup \{v_1, v_2\}]$  rooted at  $\{v_1, v_2\}$ . If  $C'$  has a rooted diamond as induced minor, then  $G \leq_{\text{im}} \text{Gem}$ .*

*Proof.* Let  $C'$  be a component of  $G \setminus S$  other than  $C$  and let  $G'$  be the graph obtained from  $G$  by:

1. applying the necessary operations (contractions and vertex deletions) to transform  $G[V(C) \cup \{v_1, v_2\}]$  into a rooted diamond;
2. deleting every vertex not belonging to  $V(C) \cup V(C') \cup \{v_1, v_2\}$ ;
3. contracting  $C'$  to a single vertex.

The graph  $G'$  is then a rooted diamond and a vertex adjacent to both endpoints of its root, that is,  $G'$  is isomorphic to  $\text{Gem}$ .  $\square$

Let us now characterize these 2-connected graphs avoiding rooted diamonds.

**Lemma 20.** *Let  $G$  be a graph rooted at  $\{u, v\} \in E(G)$ . If  $\{u, v\}$  is not a cut of  $G$  and  $G$  does not contain a rooted diamond as induced minor, then either  $G$  is an induced cycle, or both  $u$  and  $v$  are dominating in  $G$ .*

*Proof.* Assuming that  $u$  is not dominating and  $G$  is not an induced cycle, let us prove that  $G$  contains a rooted diamond as induced minor. Let  $w \in V(G)$  be a vertex such that  $\{u, w\} \notin E(G)$ . Such a vertex always exists given that  $u$  is not dominating. Let  $C$  be a shortest cycle using the edge  $\{u, v\}$  and the vertex  $w$  (which exists since  $G$  is 2-connected), let  $P_u$  be the subpath of  $C$  linking  $u$  to  $w$  without meeting  $v$  and similarly let  $P_v$  be the subpath of  $C$  linking  $v$  to  $w$  without meeting  $u$ . By the choice of  $C$ , both  $P_u$  and  $P_v$  are induced path. Notice that if there is an edge connecting a vertex of  $P_u \setminus \{w\}$  to vertex of  $P_v \setminus \{w\}$ , then  $G$  contains a rooted diamond. Therefore we can now assume that  $C$  is an induced cycle.

Recall that we initially assumed that  $G$  is not an induced cycle. Therefore  $G$  contains a vertex not belonging to  $C$ . Let  $G'$  be the graph obtained from  $G$  by contracting to one vertex  $x$  any connected component of  $G \setminus C$  and deleting all the other components. Obviously we have  $G' \leq_{\text{im}} G$ . Let us show that  $G'$  contains a rooted diamond as induced minor.

Remark that the neighborhood of  $x$ , which is of size at least two (as  $G$  is 2-connected), is not equal to  $\{u, v\}$ , otherwise  $\{u, v\}$  would be a cut in  $G$ . Now contract in  $G'$  all the edges of  $C \setminus \{u, v\}$  except three in a way such that  $|N(x)| \geq 2$  and  $N(x) \neq \{u, v\}$ . Let  $G''$  be the obtained graph, which consists of a cycle of length four rooted at  $\{u, v\}$  and a vertex  $x$  adjacent to at least two vertices of this cycle. We shall here recall that since this cycle is a contraction of the induced cycle  $C$ , it is induced too. If  $x$  is adjacent (among others) to two vertices at distance two on this cycle, then by contracting the edge between  $x$  and one of these vertices we get a rooted diamond. The remaining case is when  $x$  is only adjacent to the vertices of the cycle which are not  $u$  and  $v$ . The contraction of the edge between  $v$  and one of these vertices gives a rooted diamond, and this concludes the proof.  $\square$

*Remark 10.* In a Gem-induced minor-free graph  $G$ , every induced subgraph  $H$  dominated by a vertex  $v \in V(G) \setminus V(H)$  is a cograph.

Indeed, assuming that  $H$  is not a cograph, let  $P$  be a path on four vertices which is subgraph of  $H$ . Then  $G[V(P) \cup \{v\}]$  is isomorphic to Gem, a contradiction.

**Lemma 21.** *If  $G$  has a  $K_2$ -cut  $S = \{v_1, v_2\}$ , then every connected component of  $G \setminus S$  is basic in  $G$ .*

*Proof.* By Lemma 19, for every connected component  $C$  of  $G \setminus S$  we know that the graph  $G[V(C) \cup S]$  rooted at  $\{u, v\}$  contains no rooted diamond. By the virtue of Lemma 20, this graph either is an induced cycle, or has a dominating vertex among  $u$  and  $v$ . In the first case,  $C$  is a path whose all internal vertices are of degree two in  $G$ , hence  $H$  is basic. If one of  $u$  and  $v$  is dominating, then  $C$  is a cograph according to Remark 10. Therefore in both cases  $C$  is basic in  $G$ .  $\square$

Let us now focus on 2-connected graphs with a  $\overline{K_2}$ -cut, which is the last case in our characterization theorem.

**Corollary 8.** *If  $G$  has a  $\overline{K_2}$ -cut  $S$  such that  $G \setminus S$  contains more than two connected components, then every connected component of  $G \setminus S$  is basic in  $G$ .*

*Proof.* It follows directly from [Lemma 21](#). Indeed, if the connected components of  $G \setminus S$  are  $C_1, C_2, \dots, C_k$ , let us contract  $C_1$  to an edge between the two vertices of  $S$ . The obtained graph fulfills the assumptions of [Lemma 21](#):  $S$  is a  $K_2$ -cut. Therefore each of the components  $C_2, \dots, C_k$  is basic in  $G$ . Applying the same argument with  $C_2$  instead of  $C_1$  yields that  $C_1$  is basic in  $G$  as well.  $\square$

**Lemma 22.** *Let  $S = \{u, v\}$  be a  $\overline{K_2}$ -cut, such that  $G \setminus S$  has only two connected components  $H_1$  and  $H_2$ . Then  $G$  contains a cycle  $C$  as induced subgraph such that every connected component of  $G \setminus C$  is basic in  $G$ .*

*Proof.* For every  $i \in \{1, 2\}$ , let  $Q_i$  be a shortest path linking  $u$  to  $v$  in  $G[V(H_i) \cup \{u, v\}]$ . Notice that the cycle  $C = G[V(Q_1) \cup V(Q_2)]$  is then an induced cycle. For contradiction, let us assume that some connected component  $J$  of  $G[V \setminus C]$  is not basic in  $G$ . By symmetry, we can assume that  $J \subset H_1$ .

Notice that since  $G$  is 2-connected,  $J$  has at least two distinct neighbors  $x, y$  on  $C$ . Let  $G'$  be the graph obtained from  $G[V(H_1) \cup V(C) \cup V(H_2)]$  by contracting  $Q_1$  to an edge between  $u$  and  $v$  in a way such that  $x$  is not contracted to  $y$  (that is,  $x$  is contracted to one of  $u, v$  and  $y$  to the other one). In  $G'$ ,  $\{u, v\}$  is a  $K_2$ -cut, therefore by [Lemma 21](#), every connected component of  $G \setminus S$  is basic in  $G'$ . As this consequence holds for every choice of  $J$  and  $G'$  is an induced minor of  $G$ , we eventually get that every connected component of  $G \setminus C$  is basic in  $G$ .  $\square$

In the sequel,  $S, u, v$  and  $C$  follow the definitions of the statement of [Lemma 22](#). In order to be more accurate on how the connected components of  $G \setminus C$  are connected to  $C$ , we will prove the following lemma according to which most of the vertices of  $C$  have degree 2 in  $G$ . Let us assume that

*Remark 11.* Every connected component  $J$  of  $G \setminus C$  has at least two and at most three neighbours on  $C$ .

Indeed, it has at least two neighbours on  $C$  because  $G$  is 2-connected. Besides if  $J$  has at least four neighbours on  $C$ , then contracting in  $G[V(C) \cup V(J)]$  the component  $J$  to a single vertex, deleting a vertex of  $C$  not belonging to  $N(J)$  (which exists since  $J$  belongs to only one of the components of  $G \setminus S$ ) and then contracting every edge incident with a vertex of degree two would yield Gem.

**Lemma 23.** *If  $C$  has at least one vertex of degree two, then for every distinct connected components  $J_1$  and  $J_2$  of  $G \setminus C$  we have  $N_C(J_1) \subseteq N_C(J_2)$  or  $N_C(J_2) \subseteq N_C(J_1)$ .*

*Proof.* Let us assume, for contradiction, that the claim is not true and let  $G$  be a minimal counterexample with respect to induced minors. In such a case both  $J_1$  and  $J_2$  are single vertices (say  $j_1$  and  $j_2$  respectively) and they are the only connected components of  $G \setminus C$ . We now would like to argue that any such minimal counterexample must

contain as induced minor one of graphs presented on [Figure 12](#) (where thick edges represent the cycle  $C$ ). This would conclude the proof as each of these graphs contains Gem as induced minor, as shown in [Figure 13](#).

First of all, in such a minimal counterexample there is only one vertex in  $C$  of degree 2, let us call it  $c$ . We will consider all the ways that the vertices  $j_1$  and  $j_2$  can be connected to the neighbors of  $c$ , and show that in every such case we can contract our graph to one of the graphs on [Figure 12](#). According to [Remark 11](#), each of  $j_1$  and  $j_2$  will have either two or three neighbors on  $C$ .

*First case:* both  $j_1$  and  $j_2$  are connected with both neighbours of  $c$ . As  $N(j_1) \not\subseteq N(j_2)$  and  $N(j_2) \not\subseteq N(j_1)$ , each of  $j_1, j_2$  has a neighbor which is not adjacent to the other. But since  $j_1$  and  $j_2$  can have at most three neighbors, the neighborhood of  $j_1$  and  $j_2$  is now completely characterized. The leftmost part of [Figure 12](#) presents the only possible graph for this case.

*Second case:*  $j_1$  is connected with exactly one of neighbours of  $c$  and  $j_2$  is connected with the other one. In this case, as each of  $j_1, j_2$  has at least two neighbors on  $C$ , contracting all the edges of  $C$  whose both endpoints are at distance at least two from  $c$  gives the graph depicted in the center of [Figure 12](#).

*Third case:*  $j_1$  is connected with both neighbours of  $c$ , and  $j_2$  is connected with at most one of them. In this case, as long as  $C$  has more than 4 edges, we can contract an edge of  $C$  to find a smaller counterexample. Precisely, if there are at least 4 edges, there are two edges  $e_1, e_2$  in  $C$  within distance exactly 1 to  $c$  and those two does not share an endpoint. Moreover  $j_2$  has a neighbour  $s$  in  $C \setminus N(c)$ , say  $x$ , which is not a neighbour of  $j_1$ . Now one of the edges  $e_1, e_2$  is not incident to  $s$ , and contracting this edge would yield a smaller counterexample.

Therefore, we only have to care about the case where  $C$  has exactly 4 edges, and this case is exactly the graph represented on the right of [Figure 12](#).

We have considered all possible induced minor-minimal counterexamples (up to symmetry between  $j_1$  and  $j_2$ ). For each of these cases, which are presented on [Figure 12](#), we will now give an induced minor model of Gem, which proves that they all contain Gem as induced minor. For each graph of depicted on [Figure 13](#) we consider the model mapping the vertex  $v_i$  of Gem to the set of vertices labeled  $M_i$ . It is easy to check that each of these sets induces a connected subgraph and that the adjacencies between two sets correspond to the ones between the corresponding vertices of Gem. This concludes the proof.  $\square$

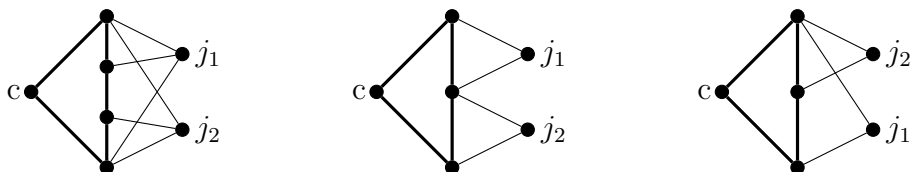


Figure 12: Induced minor-minimal counterexamples in [Lemma 23](#).

**Corollary 9.** *If  $C$  has at least one vertex of degree two, then it has at most three vertices of degree greater than two.*



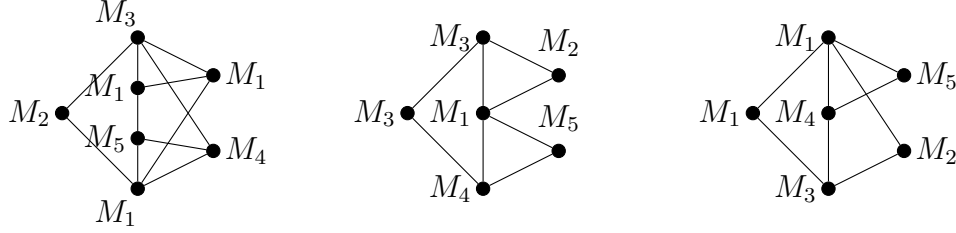


Figure 13: Models of Gem in graphs from Figure 12.

*Proof.* Notice that the set of vertices of  $C$  that have degree greater than two is exactly the union of  $N_C(J)$  over all connected components  $J$  of  $G \setminus C$ . We just saw in Lemma 23 that for every two connected components of  $G \setminus C$ , the neighborhood on  $C$  of one is contained in the neighborhood on  $C$  of the other, and that these neighborhoods have size at most three. Therefore their union have size at most three as well.  $\square$

**Corollary 10.** *Every connected component of  $G \setminus C$  is basic and  $C$  has at most six vertices of degree greater than two.*

*Proof.* Remark that contracting  $H_1$  to a single vertex  $h$  in  $G$  gives a graph  $G'$  and a cycle  $C'$  (contraction of  $C$ ) such that every connected component of  $G' \setminus C'$  is basic and  $C'$  has at least one vertex of degree 2,  $h$ . By Corollary 9,  $C'$  has at most three vertices of degree greater than two. Notice that these vertices belong to  $G' \setminus h$  which is isomorphic to  $G \setminus H_1$ . Hence  $G \setminus H_1$  has at most three vertices of degree greater than two. Applying the same argument with  $H_2$  instead of  $H_1$  we get the desired result.  $\square$

Now we are ready to prove main decomposition theorem for Gem-induced minor-free graphs.

*Proof of Theorem 3.* Recall that we are looking for a subset  $X$  of  $V(G)$  of size at most 6 such that each component of  $G \setminus X$  is basic in  $G$ .

If  $G$  is 3-connected, by Proposition 5 it is either a cograph, or has a subset  $X$  of four vertices such that every connected component of  $G \setminus X$  is a cograph. Let us now assume that  $G$  is not 3-connected.

In the case where  $G$  has a  $K_2$ -cut  $S$ , or if  $G$  has a  $\overline{K_2}$ -cut  $S$  such that  $G \setminus S$  has more than two connected components, then according to Lemma 21 and Corollary 8 respectively,  $S$  satisfies the required properties. In the remaining case, by Corollary 10  $G$  has a cycle  $C$  such that every connected component of  $G \setminus C$  is basic in  $G$  and which has at most six vertices of degree more than two in  $G$ . Let  $X$  be the set containing those vertices of degree more than two. Observe that every connected component of  $G \setminus X$  is either a connected component of  $G \setminus C$  (hence it is basic) or a part of  $C$ , i.e. a path whose internal vertices are of degree two in  $G$  (which is basic as well). As  $|X| \leq 6$ ,  $X$  satisfies the desired properties.  $\square$



## 7.2 Well-quasi-ordering of labelled cographs

We were able to show that structure of biconnected Gem-induced minor-free graphs is essentially very simple, with building blocks being cographs and long induced paths. To conclude that labelled biconnected Gem-induced minor-free graphs are wqo by induced minor relation, we will need the fact, that the building blocks, in particular labelled cographs, are themselves well-quasi-ordered by the induced minor relation.

This following result have been proven by Damaschke in [Dam90] in the unlabelled case. The proof for the labelled case follows the same general approach, we present below the sketch of the proof.

Let us denote  $\mathcal{C}$  to be a class of all cographs.

**Theorem 6.** *For any wqo  $(Q, \preceq_Q)$ , the class  $\text{lab}_{(Q, \preceq_Q)}(\mathcal{C})$  is wqo with respect to  $\leq_{\text{isg}}$ .*

*Proof.* Let us define as usual  $(S, \preceq)$  to be  $(\mathcal{P}^{<\omega} Q, \preceq^{\mathcal{P}})$ . Define  $(S^+, \preceq^+)$  such that  $S^+$  is disjoint union of  $S$  and  $\{0, 1\}$ ; the order  $\preceq^+$  is such that  $\preceq^+$  is just  $\preceq$  when restricted to  $S$ , but 0, 1 and elements of  $S$  are incomparable.

By virtue of Remark 1 and Corollary 1, we know that  $(S^+, \preceq^+)$  is wqo. By the labelled version of the famous Kruskal theorem (see [Kru60]), the class of all finite trees labelled by  $(S^+, \preceq^+)$  is wqo, with respect to a topological minor relation. In particular, we can consider class  $\mathcal{T}$  of finite trees, labelled by  $(S^+, \preceq^+)$ , such that all internal nodes have labels  $\{0, 1\}$ , and all leaves has label from  $S$ . We will consider this class again with the ordering by a labelled topological minor relation. As it is a subclass of a wqo, the class  $\mathcal{T}$  itself is also wqo. We will now provide a epi  $\phi : \mathcal{T} \rightarrow \text{lab}_{(Q, \preceq_Q)}(\mathcal{C})$ , and we will conclude by Remark 2, that  $\text{lab}_{(Q, \preceq_Q)}(\mathcal{C})$  is wqo.

The function  $\phi$  is defined as follows: given a labelled tree  $T$ , if the whole tree is only a single leaf, it produces a graph with a single vertex, and with the same label as the one the leaf has in  $T$ . If the tree is larger than a single vertex, it has root  $r$  with label  $s$ , and subtrees  $T_1, T_2, \dots, T_k$ , all rooted at some children of  $r$ . Then  $\phi(T)$  is defined as disjoint union of  $\phi(T_i)$  if label  $s$  were 0, or join of  $\phi(T_i)$  of label of  $s$  were 1. It is well-known, that every cograph has such a presentation, i.e. that function  $\phi$  indeed is surjective. The tree  $T$  which is mapped  $G$  by  $\phi$  is usually called *cotree* of  $G$ .

Now we only need to prove that  $\phi$  is monotone. Indeed, consider two trees labelled  $T_1, T_2 \in \mathcal{T}$ , such that  $T_1 \leq T_2$ , and let  $i : V(T_1) \mapsto V(T_2)$  to be an embedding of  $T_1$  in  $T_2$  as a topological minor, such that  $\lambda_{T_1}(u) \preceq^+ \lambda_{T_2}(i(u))$ . In particular, by the second property, we conclude that  $i$  maps leaves of  $T_1$  to leaves of  $T_2$ . Therefore we can consider a corresponding mapping  $\tilde{i}$  from  $V(\phi(T_1))$  to  $V(\phi(T_2))$ . Clearly it is injective, and has the property that  $\lambda_{\phi(T_1)}(u) \preceq \lambda_{\phi(T_2)}(\tilde{i}(u))$ . To show that it defines a model of  $\phi(T_1)$  as an induced subgraph of  $\phi(T_2)$ , we only need to prove that  $\tilde{i}(u)$  and  $\tilde{i}(v)$  are connected by an edge iff  $u$  and  $v$  are connected by an edge.

Remark that two vertices  $u, v \in V(\phi(T))$  are connected by an edge in  $\phi(T)$  iff the label of a lowest common ancestor of the corresponding leaves in  $T$  is 1. But  $i$  was an embedding of  $T_1$  in  $T_2$  as a topological minor, so in particular lowest common ancestor of  $i(u)$  and  $i(v)$  is the same as an image of lowest common ancestor of  $u, v$ . Moreover, by the definition of the order  $S^+$ , embedding  $i$  preserves exactly labels of internal nodes. Hence  $\phi$  is indeed monotone, and this concludes the proof of the theorem.  $\square$

### 7.3 Well-quasi-ordering Gem-induced minor-free graphs

In this section we will give a proof of [Theorem 5](#).

We define  $\mathcal{B}$  as the class of graphs which are disjoint unions of induced paths and cographs, and  $\mathcal{C}$  the class of cographs.

**Lemma 24.** *Let  $k \in \mathbb{N}$ , let  $(S, \preceq)$  be a wqo and let  $\mathcal{G}$  be the class of  $(S, \preceq)$ -labelled graphs such that the removal of at most  $k$  vertices yields a graph of  $\mathcal{B}$ . Then  $(\mathcal{G}, \leq_{\text{im}})$  is a wqo.*

*Proof.* Let  $k$  be fixed.

For every  $G \in \mathcal{G}$ , let  $X_G$  be a set of at most six vertices of  $G$  such that  $G \setminus X \in \mathcal{B}$ .

For every graph  $H$  on at most six vertices, let  $\mathcal{G}_H = \{G \in \mathcal{G}, G[X_G] = H\}$ . Observe that this gives a partition of  $\mathcal{G}$  into a finite number of subclasses. By the virtue of [Remark 1](#), we only need to focus on one of these classes. For the sake of simplicity, we assume that  $H$  has exactly  $k$  vertices  $\{v_1, \dots, v_k\}$ .

Informally, our goal is now to define a function  $f$  which constructs a graph of  $\mathcal{G}_H$  given an encoding in terms of graphs of  $\mathcal{B}$ . We will then show that  $f$  is an epi.

Let  $f$  be the function whose domain is the quasi-order

$$(\mathcal{D}, \preceq_{\mathcal{D}}) = (S, \preceq)^k \times (\mathcal{P}^{<\omega}(\text{lab}_{(S, \preceq)}(\mathcal{OP})), \leq_c^*)^{\binom{k}{2}} \times (\text{lab}_{(S, \preceq) \times (2^{[1, k]}, =)}(\mathcal{C}), \leq_{\text{isg}})$$

(where  $\preceq_{\mathcal{D}} = \preceq^k \times (\leq_c^*)^{\binom{k}{2}} \times \leq_{\text{isg}}$ ) and which, given a tuple  $((s_i)_{i \in [1, k]}, (L_{i,j})_{i,j \in [1, k], i < j}, J)$  where  $(s_i)_{i \in [1, k]} \in S^k$  is a tuple of  $k$  labels from  $S$ ,  $(L_{i,j})_{i,j \in [1, k], i < j}$  is a tuple of  $\binom{k}{2}$  subsets of  $(S, \preceq)$ -labeled oriented paths and  $J \in \text{lab}_{(S, \preceq) \times (2^{[1, k]}, =)}(\mathcal{C})$ , is a  $(S, \preceq) \times (2^{[1, k]}, =)$ -labeled cograph, returns the graph constructed as follows, starting from  $H$ :

1. label  $s_i$  the vertex  $v_i$ , for every  $i \in [1, k]$ ;
2. for every  $i, j \in [1, k]^2$ ,  $i < j$ , and for every path  $L \in L_{i,j}$ , add a copy of  $L$  to the current graph, connect  $v_i$  to  $\text{fst}(L)$  and  $v_j$  to  $\text{lst}(L)$ ;
3. add to the current graph a copy of the underlying graph of  $J$  and, for every vertex labeled  $(s, \{e_1, \dots, e_l\})$  (for some  $l \in [1, k]$ ), give the label  $s$  to the corresponding vertex in the current graph and make it adjacent to vertices  $v_{e_1}, \dots, v_{e_l}$ .

By construction, the codomain of  $f$  is included in  $\text{lab}_{(S, \preceq)}(\mathcal{G}_H)$ . Let us now show that  $f$  is surjective on  $\text{lab}_{(S, \preceq)}(\mathcal{G}_H)$ . Let  $G \in \text{lab}_{(S, \preceq)}(\mathcal{G}_H)$  and let us consider the connected components of  $G \setminus X_G$ . Let  $J$  be the disjoint union of all such components that are cographs. Note that  $J$  is a cograph as well. For every vertex  $v$  of  $J$  of label  $s$ , we relabel  $v$  with the label  $(s, \{e_1, \dots, e_l\})$ , where  $\{e_1, \dots, e_l\}$  are all the integers  $i \in [1, k]$  such that  $v$  is adjacent to  $v_i$ . For every  $i, j \in [1, k]$ ,  $i < j$ , let  $L_{i,j}$  be the set of paths of  $G \setminus X_H$  which are neighbors in  $G$  of  $v_i$  and  $v_j$ , to which we give the following orientation: the first vertex of such a path is the one which is adjacent to  $v_i$  and its last vertex is the one adjacent to  $v_j$ . Last, let  $s_i$  be the label of  $v_i$  for every  $i \in [1, k]$ . Then it is clear that  $G$  is isomorphic to  $f(\{(s_i)_{i \in [1, k]}, \{L_{i,j}\}_{i,j \in [1, k], i < j}, J)$ . Consequently  $f$  is surjective on  $\text{lab}_{(S, \preceq)}(\mathcal{G}_H)$ .

Our current goal is now, in order to show that  $f: (\mathcal{B}, \preceq_{\mathcal{B}}) \rightarrow (\text{lab}_{(S, \preceq)}(\mathcal{G}_H), \leq_{\text{im}})$  is an epi, is to prove that it is monotone. Let  $A, B$  be two elements of  $S^k \times$

$(\mathcal{P}^{<\omega}(\text{lab}_{(S,\preceq)}(\mathcal{OP})))^{\binom{k}{2}} \times \text{lab}_{(S,\preceq) \times (2\llbracket 1,k \rrbracket, =)}(\mathcal{C})$  such that  $A \preceq_{\mathcal{D}} B$ . Let us show that  $f(A) \leq_{\text{im}} f(B)$ . According to [Remark 3](#), it is enough to focus on the cases where  $A$  and  $B$  differ by only one coordinate.

*First case:*  $A$  and  $B$  differ by the  $i$ -th coordinate, for  $i \in \llbracket 1, k \rrbracket$ . Let  $s_A$  (resp.  $s_B$ ) be the value of the  $i$ -th coordinate of  $A$  (resp. of  $B$ ). According to the definition of  $f$ , the graphs  $f(A)$  and  $f(B)$  differ only by the label of vertex  $v_i$ : this label is  $s_A$  in  $f(A)$  whereas it equals  $s_B$  in  $f(B)$ . But since we have  $s_A \preceq s_B$  (as  $A \leq_B B$ ), we get  $f(A) \leq_{\text{im}} f(B)$ .

*Second case:*  $A$  and  $B$  differ by the last coordinate. Let  $J_A$  (resp.  $J_B$ ) be the value of the last coordinate of  $A$  (resp. of  $B$ ). As previously,  $A \preceq_B B$  gives  $J_A \leq_{\text{isg}} J_B$ , therefore we can obtain  $J_A$  by removing vertices of  $J_B$  and contracting labels. As the adjacencies of vertices of  $J_A$  and  $J_B$  to the rest of  $f(A)$  and  $f(B)$  (respectively) depends only on the label of their vertices, the same deletion and contraction operations in  $f(B)$  give  $f(A)$ , hence  $f(A) \leq_{\text{im}} f(B)$ .

*Third case:*  $A$  and  $B$  differ by the  $i$ -th coordinate, for some  $i \in \llbracket k+1, k + \binom{k}{2} \rrbracket$ . Let  $L_A$  (resp.  $L_B$ ) be the value of this coordinate in  $A$  (resp. in  $B$ ). As previously again,  $A \preceq_B B$  gives  $L_A \leq_c L_B$ , consequently we can obtain  $L_A$  by contracting edges of  $J_B$  and contracting labels. Since the contraction relation on  $\mathcal{OP}$  requires that endpoints (beginning and end of a path) are preserved, the same contraction operations in  $f(B)$  give  $f(A)$ , thus we again get  $f(A) \leq_{\text{im}} f(B)$ .

We just proved that  $f$  is monotone, therefore it is an epi. By [Remark 2](#), it is enough to show that  $(\mathcal{B}, \preceq_B)$  is a wqo in order to prove that  $(\text{lab}_{(S,\preceq)}(\mathcal{G}_H), \leq_{\text{im}})$  is a wqo.

Notice  $(\mathcal{B}, \preceq_B)$  is a Cartesian product of wqos and of the set of finite subsets of a wqo. Indeed, we assumed that  $(S, \preceq)$  is a wqo. Furthermore, we proved in [Lemma 17](#) that for every wqo  $(S, \preceq)$ , the quasi-order  $(\text{lab}_{(S,\preceq)}(\mathcal{OP}), \leq_c)$  is a wqo, and hence so is  $(\mathcal{P}^{<\omega}(\text{lab}_{(S,\preceq)}(\mathcal{OP})), \leq_c^*)$  (cf. [Corollary 1](#)). Last, we proved in [Theorem 6](#) that the class of cographs labeled by a wqo is well-quasi-ordered by the induced subgraph relation. Therefore,  $(\mathcal{B}, \preceq_B)$  is a wqo, which concludes the proof.  $\square$

*Proof of Theorem 5.* According to [Proposition 1](#), it is enough to prove that for every wqo  $(S, \preceq)$ , the class of  $(S, \preceq)$ -labeled 2-connected graphs which does not contain Gem as induced minor is well-quasi-ordered by induced minors. By [Theorem 3](#), these graphs can be turned into a disjoint union of paths and cographs by the deletion of at most six vertices. A consequence of [Lemma 24](#) (for  $k = 6$ ), these graphs are well-quasi-ordered by induced minors and we are done.  $\square$

## 8 Concluding remarks

In this paper we characterized all graphs  $H$  such that the class of  $H$ -induced minor-free graphs is a well-quasi-order with respect to the induced minor relation. This allowed us to identify the *boundary graphs* (Gem and  $\hat{K}_4$ ) and to give a dichotomy theorem for this problem. Our proof relies on two decomposition theorems and a study of infinite antichains of the induced minor relation. This work can be seen as the induced minor counterpart of previous dichotomy theorems by Damaschke [[Dam90](#)] and Ding [[Din92](#)].

The question of characterizing ideals which are well-quasi-ordered can also be asked for ideals defined by forbidding several elements. To the knowledge of the authors, very little is known on these classes for the induced minor relation, and thus their investigation could be the next target in the study of induced minors ideals. Partial results have been obtained when considering the induced subgraph relation [KL11a, KLR13].

## References

- [AL14] Aistis Atminas and Vadim V. Lozin. Labelled induced subgraphs and well-quasi-ordering. *Order*, pages 1–16, 2014.
- [Che11] Gregory Cherlin. Forbidden substructures and combinatorial dichotomies: Wqo and universality. *Discrete Math.*, 311(15):1543–1584, August 2011.
- [Dam90] Peter Damaschke. Induced subgraphs and well-quasi-ordering. *Journal of Graph Theory*, 14(4):427–435, 1990.
- [Din92] Guoli Ding. Subgraphs and well-quasi-ordering. *J. Graph Theory*, 16(5):489–502, November 1992.
- [Din98] Guoli Ding. Chordal graphs, interval graphs, and wqo. *Journal of Graph Theory*, 28(2):105–114, 1998.
- [Din09] Guoli Ding. On canonical antichains. *Discrete Mathematics*, 309(5):1123 – 1134, 2009.
- [DRT10] Jean Daligault, Michael Rao, and Stéphan Thomassé. Well-quasi-order of relabel functions. *Order*, 27(3):301–315, 2010.
- [FHR09] Michael R Fellows, Danny Hermelin, and Frances A Rosamond. Well-quasi-ordering bounded treewidth graphs. In *Proceedings of IWPEC*, 2009.
- [Hig52] Graham Higman. Ordering by divisibility in abstract algebras. *Proceedings of the London Mathematical Society*, s3-2(1):326–336, 1952.
- [HL14] Chun Hung Liu. *Graph Structures and Well-Quasi-Ordering*. PhD thesis, Georgia Tech, 2014.
- [iO08] Sang il Oum. Rank-width and well-quasi-ordering. *SIAM Journal on Discrete Mathematics*, 22(2):666–682, 2008.
- [KL11a] Nicholas Korpelainen and Vadim Lozin. Two forbidden induced subgraphs and well-quasi-ordering. *Discrete Mathematics*, 311(16):1813 – 1822, 2011.
- [KL11b] Nicholas Korpelainen and Vadim V. Lozin. Bipartite induced subgraphs and well-quasi-ordering. *Journal of Graph Theory*, 67(3):235–249, 2011.
- [KLR13] Nicholas Korpelainen, Vadim V. Lozin, and Igor Razgon. Boundary properties of well-quasi-ordered sets of graphs. *Order*, 30(3):723–735, 2013.
- [KRT14] M. Kamiński, J.-F. Raymond, and T. Trunck. Multigraphs without large bonds are well-quasi-ordered by contraction. *ArXiv e-prints*, December 2014.

- [Kru60] J. B. Kruskal. Well-quasi-ordering, the tree theorem and Vazsonyi's conjecture. *Transactions of the American Mathematical Society*, 95:210–225, 1960.
- [MNT88] Jiří Matoušek, Jaroslav Nešetřil, and Robin Thomas. On polynomial time decidability of induced-minor-closed classes. *Commentationes Mathematicae Universitatis Carolinae*, 29(4):703–710, 1988.
- [Pet02] Marko Petkovšek. Letter graphs and well-quasi-order by induced subgraphs. *Discrete Mathematics*, 244(1–3):375 – 388, 2002. Algebraic and Topological Methods in Graph Theory.
- [Pon91] I.N. Ponomarenko. The isomorphism problem for classes of graphs closed under contraction. *Journal of Soviet Mathematics*, 55(2):1621–1643, 1991.
- [RS93] Neil Roberston and Paul D. Seymour. *Graph Structure Theory: Proceedings of the Joint Summer Research Conference on Graph Minors, Held June 22 to July 5, 1991, at the University of Washington, Seattle, with Support from the National Science Foundation and the Office of Naval Research*, volume 147. AMS Bookstore, 1993.
- [RS04] Neil Robertson and Paul D. Seymour. Graph Minors. XX. Wagner's conjecture. *Journal of Combinatorial Theory, Series B*, 92(2):325 – 357, 2004.
- [Tho85] Robin Thomas. Graphs without  $K_4$  and well-quasi-ordering. *Journal of Combinatorial Theory, Series B*, 38(3):240 – 247, 1985.